
Exact solutions for driven-diffusive systems: Excess-mass formation and phase transitions

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Abstract A totally asymmetric exclusion process on a ring is investigated in which particles can move one or two sites. Special attention is spent to the high-speed case where particles are not allowed to move a single site if they could move two sites. The stationary state is calculated exactly in the framework of the matrix-product ansatz. Independently of the update this process evolves into subspace of the configuration space and can lead to the formation of an excess hole. One observes two phases where its velocity takes different values which are calculated exactly from the normalization-generating function. Numerically computed density profiles show an interesting algebraic form as a limit of a shock profile. For continuous time the process turns out to be related to the ASEP with a single defect particle and for synchronous update it leads to a natural defect dynamics. For the general definition of the process with maximum velocity two, from an exact analysis of the two-particle sector we find that the distribution of inter-particle distance can oscillate. This is underlined in the thermodynamic limit by an improved mean-field theory which shows a remarkably good agreement with simulations. The main focus of this work being the exact solution of lattice models by matrix-product ansatz we obtain stationary states also for related processes. Especially considerable is a new formulation of steady states for parallel dynamics like the ASEP with open boundaries as a product of a pair-factorized and a matrix-product state.

1 Preface

1.1 Introduction

A lot of attention in non-equilibrium statistical mechanics is focused on particle models on a lattice [1]. So-called ‘driven-diffusive systems’ have the property that they evolve microscopically under conservative dynamical rules. Of special interest is the (one-dimensional) asymmetric simple exclusion process (ASEP) in which not more than one particle can occupy a single lattice site (particle exclusion) and where the dynamics is asymmetric (particle movement to the right). The ASEP has been used to model for example vehicular traffic on a highway, granular flow and biological processes. For an overview of recent work see [2–6]. The usual physical time-evolution is in continuous time. This is realized in computer simulations by a random-sequential update in which a randomly chosen pair of neighboring sites is updated per time. However there are other types of update where time evolves in discrete steps. The probably most important discrete-time update is the parallel update in which all the sites are updated simultaneously and particles move with probability p . The introduction of such a parameter is necessary to interpolate stochastically between purely deterministic movement ($p = 1$) like in usual cellular automata [7] and the continuous-time limit ($p \rightarrow 0$). The correlations of the parallel update give a more realistic description of traffic [8]. However there are other discrete-time updates. For example a shuffled update [9, 10] for the ring geometry, in which the N particles are updated in a random order $\pi_N(t)$ at each time-step t . This update is used to avoid conflicts in two dimensions [11] as a model for pedestrian dynamics [12]. Ordered-sequential updates moving through the system from one end to the other have been considered as well as sublattice-parallel updates (alternatingly on the sublattices with even and odd site index), for a review see [13].

Beyond the interest of modeling many-particle systems in one dimension and giving the possibility of effective Monte-Carlo simulations [14], the ASEP has a purely theoretical interest in the context of non-equilibrium statistical mechanics. The asymmetric dynamics leads to a non-vanishing flow even in the steady state what makes it be a non-equilibrium steady state [15]. In other words, the system is driven away from equilibrium. In contrast to the equilibrium case one observes phase transitions already in one dimension [16]. Since for a given non-equilibrium system with certain microscopic rules the steady-state distribution of particles on the lattice is in general un-known, there is a large interest in exact solutions for special driven-diffusive systems. Note that for a lattice gas in equilibrium [17, 18] one can write down the statistics of the microstates in terms of the Gibbs-Boltzmann distribution.

The type of exact solutions for driven-diffusive systems are mainly factorized steady states [19], where correlations between neighboring sites are absent. The first initial breakthrough to an exact formulation of a more complex steady state was the exact solution of the ASEP with open boundaries. In this model particles enter the system on the first site and leave it from the endmost site at certain rates. It was shown that the weight for an arbitrary lattice configuration can be obtained recursively from results of smaller systems [20]. This could be written as a so-called matrix-product state, where the weights are written as

a matrix element over a product of matrices [21]. Using this formulation it is rather straightforward to calculate thermodynamical relevant quantities [15]. It showed that the ASEP with open boundaries can be in three different phases depending on the choice of boundary rates and the phase transitions have been calculated exactly. Later the matrix-product method was generalized to calculate the steady state for the ASEP on a ring with two species of particles [22, 23] which also has a nontrivial steady state exhibiting phase transitions. Since then a lot of generalizations of these processes have been investigated [15]. The open-boundary ASEP has been solved for parallel update too, but since then there is somehow a lack of exact solutions. A reason might be that the matrix product for parallel dynamics in its original versions [24, 25] took a rather difficult form. A short but important result of this thesis is the reformulation of this solution as a product of a pair-factorized and a matrix-product state. The pair-factorized state appears also for periodic boundaries [26] and arises from nearest-neighbor correlations (a particle-hole attraction); the matrix-product state takes a form comparable to the states for other discrete-time dynamics, such as ordered-sequential updates [27] and sublattice-parallel updates [28, 29]. In this work we mainly consider a generalization of the ASEP on a ring where (identical) particles can move either one or two sites to the right; overtaking is forbidden and particles obey the exclusion rule (at most one particle per site) [30–32]. The time evolution can lead to the formation of what will be referred to as ‘excess mass’ in analogy to a quantity of interest in the mathematical literature [33]. The original motivation to study this model was the question: Which is the simplest totally asymmetric exclusion model with non-trivial stationary state that is exactly solvable. Non-trivial means here not factorizable [34] into one- or two-site factors or into factors for each cluster of consecutive particles or unoccupied sites [35]. The concept of factorizing the steady state into products for local n -site clusters [26] is helpful as an approximation technique but is not straightforwardly used to describe non-trivial models exactly. In fact most of the ASEP-like models defined by some local update rule have a more complex structure as a product of three- or four-site factors. However some progress has been made on pair-factorization [36] for models with infinite state space. It will turn out that the ASEP with excess-mass formation considered here is in some sense the next-simple solvable process between factorizing exclusion processes and unsolvable models. A reason is that the (for these processes typical) distribution of inter-particle distance is thermodynamically the same as the prediction from mean-field. However if one wants to know the whole structure of the stationary state one has to activate a rather complex mathematical formalism. The excess mass turns out to play the role of a defect: The defect ASEP was first introduced and studied for the case where all the hop rates are the same in which it is referred to as a second-class particle [37]. Since in an environment of particles (holes) it can only move to the left (right) it always finds positions with positive density gradient ($0 \dots 0.21 \dots 1$). With the help of the exact solution one was then able to calculate the density profile in the frame of the second-class particle. This has been considered as a limiting case of a shock profile. A shock is defined as a sudden change in the density approaching two different values to the left and right however since on a ring the density is constant, the density seen from the second-class particle far to the left and right is the same. Therefore it is a limit of a shock profile. However one can recover the full shock profile by introducing several second-class particles [22]. This shock can also be found in one phase of the defect ASEP and in the open-boundary ASEP along the second-order transition line [4]. Originally it described shocks with the same profile in the ASEP on the infinite line [38, 39]. For a recent review on one-dimensional driven-diffusive systems with two species of particles see [40]. In the ASEP with excess-mass formation we find two phases. In the one phase the profiles are algebraic and can also for parallel

dynamics be thought of as limiting shock profiles. Note that already in 1993 shock profiles in lattice fluids with parallel update have been studied [41]. However to our knowledge, a general investigation from the viewpoint of a defect as for continuous time is still missing. The modeling of traffic with the help of cellular automata goes back to 1992 in [8], later usually referred to as Nagel-Schreckenberg model. To reproduce the typical flow-density relations, which are called fundamental diagrams, it was necessary to introduce a higher maximum velocity. In this sense the process with general hop rates that we investigate here is a simplistic rather than a minimal traffic model. In contrast to the Nagel-Schreckenberg model it has no temporal memory: The probability that a particle moves one or two sites does not depend on the number of sites it has moved in the time-step before. However this memory is necessary for a physical acceleration and breaking; especially if one considers higher maximum velocities. For the simplistic model we will find that the distribution of headway (the number of empty sites ahead) can oscillate which has no relevance for real traffic but is of theoretical interest. Obviously a factorized state corresponds to physical – exponentially decaying – distributions. The ASEP with two species of particles has another natural interpretation as a toy model for traffic: The particles of the second species can be thought of as trucks in an environment of cars [4]. If cars can not overtake the truck this leads to a condensate that has an exact analogy to Bose-Einstein condensation [42]. By interchanging the definitions of trucks and unoccupied sites it models also two-way traffic, where trucks move in the opposite direction [43, 44]. With the behavior being quite universal it serves also as a model for more involved models of two-way traffic [43, 45].

1.2 Outline

The outline is as follows. In chapter 2 a short introduction is given into the mathematical framework and the relevant driven-diffusive systems known from the literature.

Chapter 3 deals with the ASEP with excess-mass formation and random-sequential dynamics. The steady state is calculated exactly with a generalization of the matrix-product ansatz to infinite-state space. Here it turns out that surprisingly the state is factorizable in one case since the process decouples into two different processes depending on the number of particles and sites. The steady state is proven in section 3.1 with the help of the canceling mechanism. The relevant matrix algebra is interpreted in 3.2 as a recursion relating stationary weights with different particle number and lattice size. Section 3.3 shows that under the usual matrix-product ansatz one cannot find a unified matrix representation for the process as a consequence of the decoupling into different subspaces of the configuration space. In the case where the state is of matrix form the process is mapped onto the ASEP with a single defect particle in section 3.4 – a relation that is not obvious. Section 3.5 then gives a unified formulation of the solution by change of the ansatz – three different possibilities are presented. Relevant quantities as the canonical normalizations, the velocity and flow as well as density correlations are calculated in 3.6. Section 3.7 is dedicated to the case of a finite number of particles on a ring of arbitrary size. The effects of adding particles on the headway distribution are discussed and scaling arguments are given. The phase transition, discussed in section 3.8, is an application of a similar transition in the defect ASEP that has not the typical jammed-fluid phase interpretation but just separates different defect velocities and density profiles. Since the process belongs to a large class of mass-transport models (section 3.9) known in the literature, the occurrence of a factorized steady state is explained in the general context and some statements on the exactness of distributions for non-factorizing processes are given.

In chapter 4 a simpler formulation of the weights of the ASEP with open boundaries and parallel dynamics is derived and the connection with the site- and bond-oriented solutions are given.

This prepares the matrix solution of the ASEP with excess-mass formation for parallel dynamics in chapter 5. The proof of the steady state is given in section 5.2. The tour de force starts with a derivation of the master equation from the local dynamical rules. The final expression of the master equation is then proven by case differentiation rather than in [30] by a more general proof. The normalization-generating function is derived in detail in section 5.3 which is verified on known limits in section 5.4. The normalization then can be used (section 5.5) to obtain asymptotic quantities such as occupations around the excess mass and its velocity in dependence on the global density. A phase diagram is derived that separates two different regimes. Numerical density profiles underline the phase behavior by mass-density distributions seen from the excess mass. A partially deterministic case is figured out in section 5.6 that can be solved with a simpler matrix representation and shows interestingly oscillating density profiles. Section 5.7 concludes with an outlook on further research.

Chapter 6 generalizes the ASEP with excess-mass formation by allowing particles to move one site if it would be possible to move two sites. This is referred to as the ASEP with maximum velocity two. Two simple special cases have been studied before which are the starting point in section 6.1. As it turned out for the ASEP with excess-mass formation, a formulation in terms of mass variables (section 6.2) is more appropriate for an analytic investigation. Section 6.3 obtains the exact steady state for two particles on a ring of arbitrary size which can lead to oscillations in the headway distribution. A case is pointed out

where the weights are given by Fibonacci numbers. Section 6.4 gives a mean-field theory for the thermodynamic limit in terms of the mass variables. By a detailed analysis of the generating function the theory predicts mass distributions for arbitrary hop rates. However the focus is on special cases again. The Fibonacci case shows a remarkable agreement with simulations. Further the theory predicts correctly the thermodynamic single-particle headway distribution of the ASEP with excess-mass formation. Section 6.5 gives a note on the process with parallel dynamics. However a detailed description is spared out since the main effects are contained already in the continuous-time case.

Beyond that we obtain the exact form of the steady state for related driven-diffusive systems without particle exclusion that also exhibit the formation of excess mass. These solutions are presented in chapter 7. They include a two-species zero-range process with matrix product state (section 7.1), whose steady state with a single-defect is related to the ASEP with excess-mass formation. Section 7.2 mentions shortly a process related to the ASEP with excess-mass formation with uniform state. Section 7.3 considers different generalizations of the zero-range process that are formally solvable in terms of a matrix product. Finally section 7.4 studies a three-state process that also leads to formation of excess mass.

2 Driven-diffusive systems

Driven-diffusive systems, as considered here, are stochastic processes defined on a one-dimensional lattice comprising L sites, labeled from the left to the right $l = 1, 2, \dots, L$. Each site can be in either of $S + 1$ states. We think of the sites being occupied by s particles ($s = 0, 1, \dots, S$). The movement of particles is completely asymmetric, i.e. they are transferred only in one direction. Beyond that the dynamics is conservative (no creation and annihilation of particles).

2.1 Mathematical form of the steady states

2.1.1 Master equation and steady state

The state of the system is defined by the set of all occupations s_l , i.e. the number of particles at each site. Thus one can write for a configuration $\mathcal{C} = \{s_1, s_2, \dots, s_L\}$. Transition rates between different states, say from \mathcal{C} to \mathcal{C}' are written as $\omega(\mathcal{C} \rightarrow \mathcal{C}')$. The probability that the system is in state \mathcal{C} at time t is denoted by $P(\mathcal{C}, t)$. The time evolution of these probabilities is governed by the so-called master equation:

$$\frac{\partial}{\partial t} P(\mathcal{C}, t) = \sum_{\mathcal{C}'} P(\mathcal{C}', t) \omega(\mathcal{C}' \rightarrow \mathcal{C}) - \sum_{\mathcal{C}'} P(\mathcal{C}, t) \omega(\mathcal{C} \rightarrow \mathcal{C}'). \quad (2.1)$$

It simply means that the probability to find configuration \mathcal{C} at time t changes with time due to transitions into state \mathcal{C} (the ‘gain’ term – the first sum on the right-hand side) and due to transitions from state \mathcal{C} into any other state (the loss term – second sum on the right-hand side). By the knowledge of $P(\mathcal{C}, t)$ for all \mathcal{C} and t the time evolution is known completely.

Here we are interested only in the long-time limit, where the system reaches a state in which the probability for configurations is temporally conserved. This is referred to as ‘steady state’. The existence and ubiquity of such a steady state usually is well-defined for the type of driven-diffusive systems considered here, since they are typically Markov processes for which the Perron-Frobenius Theorem [46] holds. In the steady state the probabilities become independent of time: $P(\mathcal{C}, t) \rightarrow P(\mathcal{C})$ and are referred to as ‘stationary probabilities’. Then (2.1) simplifies to

$$\sum_{\mathcal{C}'} P(\mathcal{C}') \omega(\mathcal{C}' \rightarrow \mathcal{C}) = \sum_{\mathcal{C}'} P(\mathcal{C}) \omega(\mathcal{C} \rightarrow \mathcal{C}'). \quad (2.2)$$

In other words, gain and loss terms are equal. In order to obtain the exact steady state for a certain driven-diffusive system in terms of the probabilities $P(\mathcal{C})$, one has to solve this type of equation. This is one of the major topics of this thesis. There are different mathematical forms of steady states that have been obtained exactly for different driven-diffusive systems [15]. These are presented in the following.

Uniform steady state

This is the simplest form of a steady state. All configurations are equally probable,

$$P(\mathcal{C}) = \text{const.} \quad (2.3)$$

In this case one has no correlations between the particles.

Factorized steady state

Slightly more general are factorized steady states. Factorized means that the probability for a certain configuration can be written as a sequence of scalar factors for each site

$$P(s_1, s_2, \dots, s_L) \propto \prod_{l=1}^L f(s_l). \quad (2.4)$$

The $f(s)$ are so-called ‘single-site weights’. The whole expression in (2.4) has to be normalized which is indicated by the proportional sign. A consequence of a factorized steady state is the absence of correlations between adjacent sites. If the state space is such that each site can only be in either of two different states ($S = 1$) then this recovers (for a fixed number of particles and sites) the uniform steady state (2.3). However if $S \geq 2$ then the factorized solution contains also information about correlations between particles occupying the same site. Such processes are well-studied – a special interest being the investigation of condensation transitions [19].

Pair-factorized steady state

A natural generalization of the factorized steady state is the pair-factorized steady state in which one has factors for each two neighboring sites:

$$P(s_1, s_2, \dots, s_L) \propto \prod_{l=1}^L t(s_l, s_{l+1}). \quad (2.5)$$

Thus one has correlations between neighboring sites. Of course this idea could be generalized to states that factorize into three-site terms and so on. In this context it has been used as an approximation technique for higher correlated processes [26]. Recently a class of processes has been found which has this type of steady state [36].

Matrix-product state

The matrix-product technique has originally been introduced in the context of directed lattice animals [47] and spin chains [48]. Later it has been applied to the ASEP [21] and related models, for a recent review see [15]. The idea is a generalization of the factorized steady state, now writing a matrix valued factor $X(s_l)$ for each site l containing s_l particles. To obtain a scalar probability one takes some sort of matrix element of the matrix product. In principle one can write it always as a trace operation on the matrix product:

$$P(s_1, s_2, \dots, s_L) = \text{tr} \left[\prod_{l=1}^L X(s_l) (\times B) \right] \quad (2.6)$$

where $(\times B)$ indicates that in cases of non-periodic boundary conditions one has to multiply by a matrix reflecting the boundaries.

In the so-called quantum formalism for stochastic models (see e.g. [13] for a detailed explanation) one introduces a vector $|P\rangle$ containing as elements the probabilities $P(\mathcal{C})$ for all configurations \mathcal{C} in lexicographic order. This is used in the ‘grand-canonical matrix ansatz’. Define a column vector $X = (X(1), X(2), \dots, X(S))^t$. Then one has

$$|P\rangle = \text{Tr} X^{\otimes L} := \begin{pmatrix} \text{tr } X(0)^L \\ \text{tr } X(0)^{L-1} X(1) \\ \vdots \\ \text{tr } X(0)^{L-1} X(S) \\ \vdots \\ \text{tr } X(S)^L \end{pmatrix} \quad (2.7)$$

This probability vector then should be solution of the master equation, rewritten grand-canonically as a sort of Schrödinger equation $\mathcal{H}|P\rangle = 0$. The stochastic Hamilton operator \mathcal{H} is a sum of local operators $\mathcal{H} = \sum_{l=1}^L h_l$. The local operator h is defined through the stochastic dynamics. Its diagonal elements are basically $-\sum$ of escape rates from the local state. And the non-diagonal terms are the rates into this local state respectively. Using this notation there is a very formal way to prove a matrix-product state which is called the ‘canceling-mechanism’ [49]. Take for simplicity periodic boundary conditions (Then in (2.6) one has $B = \mathbb{1}$). If one can find additionally to X another vector $\bar{X} = (\bar{X}(0), \bar{X}(1), \dots, \bar{X}(S))$ such that the relation

$$hX^{\otimes 2} = \bar{X}X - X\bar{X} \quad (2.8)$$

holds, then (2.7) is solution of the master equation. This can be seen as follows [50]: The action of each h_l on $|P\rangle$ is of the form

$$h_l|P\rangle = |\pi(l)\rangle - |\pi(l+1)\rangle, \quad (2.9)$$

with

$$|\pi(l)\rangle = \text{Tr} \left[X^{\otimes(l-1)} \bar{X} X^{\otimes(L-l)} \right]. \quad (2.10)$$

Therefore:

$$\mathcal{H}|P\rangle = \sum_{l=1}^L h_l|P\rangle = \sum_{l=1}^L [|\pi(l)\rangle - |\pi(l+1)\rangle] = 0. \quad (2.11)$$

In the case of other boundary conditions the mechanism is slightly changed to assure that terms at the boundary $h(B \otimes X)$ cancel too but the idea is the same.

Tensor-product state

The matrix-product technique with factors $X(s)$ has been extended to more general algebraic operators $\mathcal{X}(s)$. Often one can think of matrices whose components itself are matrices, see e.g. [24, 51].

2.2 The asymmetric simple exclusion process (ASEP)

The ASEP was introduced as a model for the kinetics of polypeptide synthesis and was originally formulated on a d -dimensional lattice in the mathematical literature, see [17, 33, 47, 52] and references therein.

The present work deals only with the one-dimensional version of the ASEP which is a

paradigmatic model for driven diffusive systems, see [53–55]. In non-equilibrium statistical mechanics it plays a role analogue to the Ising model in equilibrium [15] due to the possibility [56] of phase transitions in one dimension. Beyond this theoretical relevance it can be used to model various physical processes: It serves as a minimal model for unidirectional traffic flow [2] on a highway for instance, as a model for bio-polymerization [57] and latterly as a model for biophysical transport especially of molecular motors [3]. The ASEP is defined on a 1d lattice with L sites enumerated from the left to the right $l = 1, 2, \dots, L$. Each of these sites l can be in either of two different states:

$$\tau_l = \begin{cases} 1, & \text{if site } l \text{ is occupied by one particle} \\ 0, & \text{if it is unoccupied.} \end{cases} \quad (2.12)$$

Thus two or more particles per site are forbidden which is the statement of the exclusion rule. Originally one applies a random sequential update which means that during each time-step dt one chooses a pair of sites $(l, l+1)$ at random. This pair of sites is actualized in the following way: If site l is occupied by a particle ($\tau_l = 1$) and site $l+1$ is empty ($\tau_{l+1} = 0$) then the particle is transferred from site l to site $l+1$ with probability dt . In all other cases nothing happens. Since there is only movement from the left to the right it is the *asymmetric* exclusion process. Sometimes one allows also for movement to the left [58] at a different rate which is then referred to as partially asymmetric exclusion process. Though remember that we will deal here only with (totally) asymmetric dynamics.

Finally one has to impose boundary conditions to the lattice. The most important types are periodic (see sec. 2.2.1) and open boundary conditions (see sec. 2.2.2). Other choices are the infinite (no boundaries) and the half-infinite lattice (one hard boundary) which we do not consider here.

2.2.1 Periodic boundary conditions

Periodic boundary conditions means that site $L+1$ of the lattice is identified with site 1. One can think of the lattice being curved to a ring. Due to the conservative dynamics of the ASEP the particle number, say N , is a conserved quantity. The form of the steady state is easily obtained [4]:

Think of an arbitrary configuration \mathcal{C} with N particles and L sites. Let us further assume that there are x ‘clusters’ in the system. A ‘cluster’ is an uninterrupted block of consecutive particles that is isolated by at least one hole to left and to the right. At first note that $\omega(\mathcal{C} \rightarrow \mathcal{C}')$ can only take values 1 or 0 (either the configuration \mathcal{C}' can be reached from \mathcal{C} by moving one particle one site to the right at rate 1, then one has $\omega(\mathcal{C} \rightarrow \mathcal{C}') = 1$ or this is impossible – then one has $\omega(\mathcal{C} \rightarrow \mathcal{C}') = 0$). If there are x clusters then \mathcal{C} can turn into x other configurations \mathcal{C}' just by moving the rightmost particle of each cluster one site to the right. On the other hand \mathcal{C} can arise from x configurations \mathcal{C}' . These configurations can be obtained in mind by moving the leftmost particle of a cluster one site to the left (and backwards in time). So one has

$$\sum_{\mathcal{C}'} \omega(\mathcal{C} \rightarrow \mathcal{C}') = \sum_{\mathcal{C}'} \omega(\mathcal{C}' \rightarrow \mathcal{C}). \quad (2.13)$$

Now one sees that for (2.2) to hold together with (2.13), the steady-state probabilities have to satisfy $P(\mathcal{C}) = P(\mathcal{C}')$ for all \mathcal{C} and \mathcal{C}' . Therefore the simple solution is that each configuration of the lattice has the same (steady-state) probability (2.3) which can directly be worked out combinatorially: The probability to find a particle somewhere is N/L . The probability to find a second somewhere else in the system is $(N-1)/(L-1)$ and so on.

The probability to find the last particle on one of the remaining sites is $1/(L - N + 1)$. The product of all this factors gives

$$P(\tau_1, \tau_2, \dots, \tau_L) = \frac{N!(L - N)!}{L!} \delta_{\sum \tau_i, N}. \quad (2.14)$$

Using such arguments one can write down all equal-time correlation functions. For example the one- and two-point functions:

$$\langle \tau_l \rangle = \frac{N}{L}, \quad \langle \tau_k \tau_l \rangle_{k \neq l} = \frac{N}{L} \frac{N - 1}{L - 1}, \dots \quad (2.15)$$

A correlation function of special interest is the flow:

$$J = \langle \tau_l (1 - \tau_{l+1}) \rangle = \frac{N}{L} \frac{L - N}{L - 1}. \quad (2.16)$$

Often one is interested in the behavior for (thermodynamically) large systems and particle number. This is the thermodynamic limit $N \rightarrow \infty$, $L \rightarrow \infty$, where the value of N/L is held fixed. This defines the particle density $\rho = N/L$. One finds using Stirling's approximation

$$P(\mathcal{C}) \rightarrow \rho^N (1 - \rho)^{L - N}. \quad (2.17)$$

This can also be written as a factorized steady state (2.4) with simple single-site probabilities (!) $f(1) = \rho$ and $f(0) = 1 - \rho$. The m -point function simply turns into ρ^m and the probability for a cluster of length m is $\rho^m (1 - \rho)^2$. The relation of the flow to the density is the so-called fundamental diagram: $J(\rho) = \rho(1 - \rho)$.

2.2.2 Open boundary conditions

The ASEP with open boundaries is of much more theoretical interest as the periodic version since its steady state is highly non-trivial but exactly solvable. Depending on the model parameters one finds different density profiles and especially one finds phase transitions that one can study exactly.

Open boundaries means that the system is in contact with two particle reservoirs at both end. At the left and particles are injected from the one reservoir and on the right end particles can be removed into the second reservoir. To be precise, the dynamics is as follows: If site 1 is empty, a particle enters on site 1 at rate α . If site L is occupied it leaves the lattice at rate β . The bulk dynamics remains unchanged – a particle moves one site to the right at rate 1 supposed that it is empty. Let us introduce unnormalized weights for each configuration: $F_L(\tau_1, \tau_2, \dots, \tau_L)$. They lead to the normalization $Z_L = \sum_{\tau_1=0,1} \sum_{\tau_2=0,1} \dots \sum_{\tau_L=0,1} F_L(\tau_1, \tau_2, \dots, \tau_L)$. Here the normalization is done without fixing the particle number since it is not a conserved quantity as for periodic boundary conditions! Derrida et. al. [20] found that the weights for system size L can be constructed recursively by the knowledge of the weights for system size $L - 1$ which we write here as

$$\begin{aligned} & F_L(\tau_1, \dots, \tau_L) \\ &= \beta^{-1} \tau_L F_{L-1}(\tau_1, \dots, \tau_{L-1}) \\ &+ (1 - \tau_L) \tau_{L-1} [F_{L-1}(\tau_1, \dots, \tau_{L-2}, 1) + F_{L-1}(\tau_1, \dots, \tau_{L-2}, 0)] \\ &+ \dots \\ &+ (1 - \tau_L)(1 - \tau_{L-1}) \dots (1 - \tau_2) \tau_1 [F_{L-1}(1, \tau_2, \dots, \tau_{L-1}) + F_{L-1}(0, \tau_2, \dots, \tau_{L-1})] \\ &+ \alpha^{-1} (1 - \tau_L)(1 - \tau_{L-1}) \dots (1 - \tau_1) F_{L-1}(\tau_1, \dots, \tau_{L-1}). \end{aligned} \quad (2.18)$$

Now let us interpret this recursion. For each configuration there remains only one nonzero term on the right hand side. In detail:

- First term on the r.h.s.
if $\tau_L = 1$: $F_L(\tau_1, \dots, \tau_{L-1}, 1) = \beta^{-1} F(\tau_1, \dots, \tau_{L-1})$,
- Second term:
if $\tau_L \neq 1$ and $\tau_{L-1} = 1$: $F_L(\tau_1, \dots, \tau_{L-2}, 1, 0) = F_{L-1}(\tau_1, \dots, \tau_{L-2}, 1) + F_{L-1}(\tau_1, \dots, \tau_{L-2}, 0)$
...,
- L th term:
if $\tau, \tau_{L-1} \dots \tau_2 \neq 1, \tau_1 = 1$: $F_L(\tau_1, 0, \dots, 0) = F_{L-1}(1, 0, \dots, 0) + F_{L-1}(0, \dots, 0)$,
- $L + 1$ th term:
if $\tau, \tau_{L-1} \dots \tau_1 \neq 1$: $F_L(0, 0, \dots, 0) = \alpha^{-1} F_{L-1}(0, 0, \dots, 0)$.

Thus one always has to replace

$$F_L(\dots 10 \dots) \quad \text{by} \quad F_{L-1}(\dots 1 \dots) + F_{L-1}(\dots 0 \dots), \quad (2.19)$$

$$F_L(\dots 1) \quad \text{by} \quad \beta^{-1} F_{L-1}(\dots), \quad (2.20)$$

$$F_L(0 \dots) \quad \text{by} \quad \alpha^{-1} F_{L-1}(\dots). \quad (2.21)$$

This mechanism suggests a ‘matrix-product ansatz’ [21] for the weights (2.6). The way one writes the solution usually is

$$F(\tau_1, \dots, \tau_L) = \langle W | \prod_{l=1}^L [\tau_l D + (1 - \tau_l) E] | V \rangle, \quad (2.22)$$

the matrices E and D representing a hole and particle respectively and the vectors $\langle W |$ and $|V\rangle$ standing for the boundaries. The mechanism (2.19-2.21) is rewritten as an ‘algebra’ for the operators:

$$DE = D + E, \quad (2.23)$$

$$\alpha \langle W | E = \langle W |, \quad (2.24)$$

$$\beta D | V \rangle = | V \rangle. \quad (2.25)$$

There are different known representations of this algebra [21].

Equal-time steady-state correlation functions can be calculated by the knowledge of the normalization Z_L . It turns out that it becomes [15]

$$Z_L = \langle W | (D + E)^L | V \rangle = \sum_{i=1}^L B_{L,i} \frac{\alpha^{-1-i} - \beta^{-1-i}}{\alpha^{-1} - \beta^{-1}}, \quad (2.26)$$

where the Ballot numbers $B_{L,i}$ are

$$B_{L,i} = \frac{i(2L - i - 1)!}{L!(L - i)!}. \quad (2.27)$$

The asymptotics of (2.26) are [59]:

$$Z_L \sim \begin{cases} N^{-3/2} 4^N, & \text{for } \alpha > 1/2 \text{ and } \beta > 1/2, \\ [\alpha(1 - \alpha)]^{-N}, & \text{for } \alpha < 1/2 \text{ and } \beta > \alpha, \\ [\beta(1 - \beta)]^{-N}, & \text{for } \beta < 1/2 \text{ and } \alpha > \beta \end{cases} \quad (2.28)$$

indicating three different phases.

- I) Maximum current phase
If both α and β exceed $1/2$ then the density in the system does not depend on α or β . It is completely determined by the bulk-hopping rate 1 and reaches its maximum value $\rho_I = 1/2$. The flow (current) takes its maximum $J_I = 1/4$.
- II) Low-density phase
This phase is observed in the parameter regime $\alpha < 1/2$ and $\beta > \alpha$. If the inflow into the system is small, and the outflow is larger than the inflow, then the density in the system is small and determined by α taking simply the value $\rho_{II} = \alpha$. The flow is accordingly $J_{II} = \alpha(1 - \alpha)$.
- III) High density phase
If equivalently $\beta < 1/2$ and $\alpha > \beta$ then the density is large and determined by $\rho_{III} = 1 - \beta$. The flow is $J_{III} = \beta(1 - \beta)$.

Figure 2.1 shows the corresponding phase diagram. The thick line denotes a first-order transition and the thin lines second-order transitions.

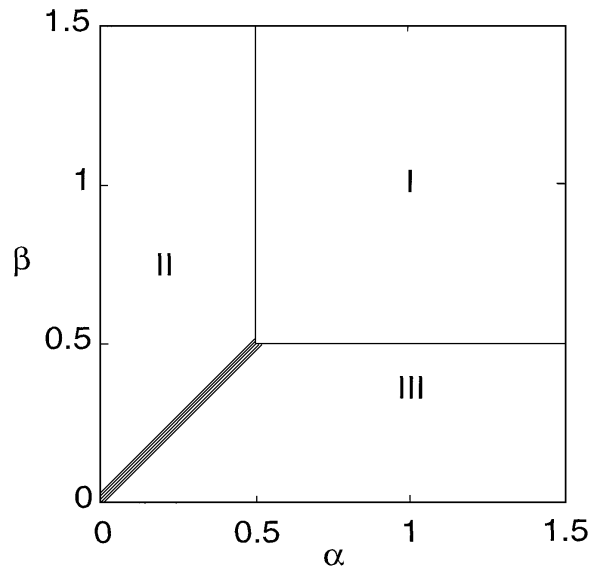


Figure 2.1: Phase diagram of the ASEP with open boundaries, taken from [4].

2.2.3 Two species of particles on a ring

A generalization of the ASEP with periodic boundary conditions to a system with one defect particle has been established for the study of shocks [22, 23]. For the defect particle one usually writes symbolically 2 instead of 1 for normal particles. The particles move according to the following rules:

$$10 \rightarrow 01, \quad \text{at rate } 1, \quad (2.29)$$

$$20 \rightarrow 02, \quad \text{at rate } \alpha, \quad (2.30)$$

$$12 \rightarrow 21, \quad \text{at rate } \beta. \quad (2.31)$$

It turns out [23] that the steady state has a matrix product form (2.6) and one can use the same matrices E , denoting a hole, and D denoting a ‘normal’ particle, if one writes a matrix A for the 2-particle or defect. Then the matrix solution reads:

$$P(2, \tau_2, \dots, \tau_L) = Z_{L,N}^{-1} \text{tr } A \prod_{l=2}^L [\tau_l D + (1 - \tau_l) E], \quad (2.32)$$

where the defect is fixed to sit on site 1. The matrix A has to be composed from the boundary vectors $\langle W|$ and $|V\rangle$ of the ASEP with open boundaries $\langle W|$ and $|V\rangle$ as follows: $A = |V\rangle\langle W|$. Therewith one can write also

$$P(2, \tau_2, \dots, \tau_L) = Z_{L,N}^{-1} \langle W| \prod_{l=2}^L [\tau_l D + (1 - \tau_l) E] |V\rangle. \quad (2.33)$$

Thus the weights can be found also in the solution of the open boundary problem (2.22) but the normalization $Z_{L,N}$ differs due to the particle conservation. The variant of the DEHP-algebra (2.23-2.25) reads here¹

$$DE = D + E, \quad (2.34)$$

$$\alpha AE = A, \quad (2.35)$$

$$\beta DA = A. \quad (2.36)$$

Having the exact form of the steady state one is able to calculate the density profile as seen from the defect particle [23].

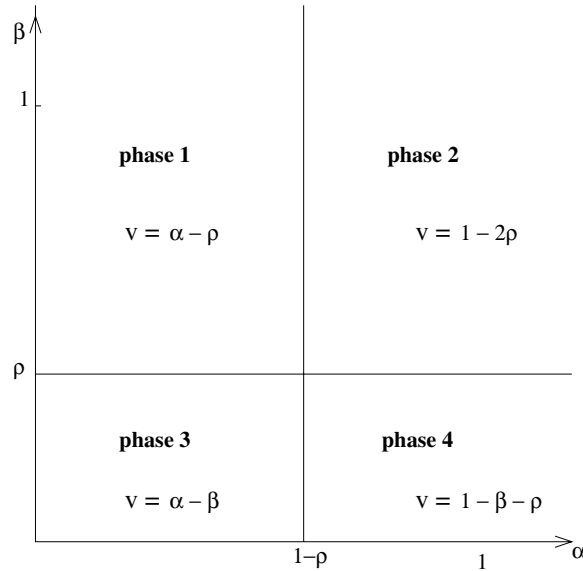


Figure 2.2: Phase diagram of the ASEP with a single defect, taken from [60].

For the case of $\alpha = \beta = 1$ the defect is referred to as a second-class particle and the corresponding density profile was argued to be a limiting shock profile (with equal densities

¹Throughout this work, these relations will often come into play. The context is always the formation of excess mass in the form of a defect in various sorts of driven-diffusive systems.

far to the left and right of the defect) [22]. The second-class particle takes positions of a positive density gradient, i.e. it prefers having particles in front and unoccupied sites to the left. Once reaching such a situation $00\dots 0211\dots 1$ it can no longer move. Thus it marks the position of a shock, since the expectation of densities to the left and right are different. For general choice of α and β one finds four phases [23] with different defect velocities v . The following table shows the average occupation on the site behind the defect ρ_- and the occupation on the site in front ρ_+ :

phase	ρ_-	ρ_+	defect behaves as
1	$p(1 - \alpha)/\beta$	ρ	a hole
2	p^2/β	$1 - (1 - \rho)^2/\alpha$	2nd-class particle
3	$1 - \alpha$	β	a moving obstacle \rightarrow shock
4	ρ	$1 - (1 - \rho)(1 - \beta)/\alpha$	a particle

In phase 3 the defect blocks particles behind it, since they can not easily overtake it. This leads to a global phase separation and the density profile presents a shock. The special case $\beta = 0$ as been considered in [42]. For later use we note that in phase 2 the defect behaves essentially as a second-class particle so that its velocity becomes independent of α and β . Since the derivatives of the velocity in the four phases are discontinuous, the phases are separated by first-order transition lines [23].

Generalizations to partially asymmetric hopping [61] have been investigated as well as generalizations to open boundaries [62] which we do not consider here.

2.2.4 The ASEP with parallel dynamics

Consider at first the ring geometry. The stationary state with weights F_L^{ring} is of the pair-factorized (or two-cluster) form [26]:

$$F_L^{\text{ring}}(\tau_1, \dots, \tau_L) = \prod_{i=1}^L t(\tau_i, \tau_{i+1}) \quad (2.37)$$

with some simple two-site factors $t(\tau_i, \tau_{i+1})$ defined through

$$\frac{t(11)}{t(10)} = (1 - p) \frac{t(10)}{t(00)}. \quad (2.38)$$

So one can set for example

$$t(11) = (1 - p)t(01) \quad \text{and} \quad t(10) = t(00). \quad (2.39)$$

The ASEP with open boundary conditions has been solved alternatively by two different versions of the matrix-product technique.

The site-oriented solution

By introducing boundary vectors $\langle W|, |V\rangle$ and matrices E and D for holes and particles respectively. Evans, Rajewsky and Speer could show that the matrix-product ansatz of the

form (2.22) gives the correct steady-state weights, when the operators satisfy the following relations. One has relations for the bulk:

$$DDEE = (1-p)DDE + (1-p)DEE + p(1-p)DE, \quad (2.40)$$

$$DDED = DDD + (1-p)DED + pDD, \quad (2.41)$$

$$EDEE = (1-p)EDE + EEE + pEE, \quad (2.42)$$

$$EDED = EDD + EED + pED, \quad (2.43)$$

as well as relations for the right boundary

$$DDE|V\rangle = (1-p)DE|V\rangle + DD|V\rangle, \quad (2.44)$$

$$EDE|V\rangle = ED|V\rangle + EE|V\rangle, \quad (2.45)$$

$$DD|V\rangle = \frac{p(1-\beta)}{\beta}D|V\rangle, \quad (2.46)$$

$$ED|V\rangle = \frac{p}{\beta}E|V\rangle, \quad (2.47)$$

and left boundary

$$\langle W|DEE = (1-p)\langle W|DE + \langle W|EE, \quad (2.48)$$

$$\langle W|DED = \langle W|DD + (1-p)\langle W|ED, \quad (2.49)$$

$$\langle W|EE = \frac{p(1-\alpha)}{\alpha}\langle W|E, \quad (2.50)$$

$$\langle W|ED = \frac{p}{\beta}\langle W|D. \quad (2.51)$$

Note that – as a consequence of the particle-hole symmetry of the process – these relations are invariant under exchanging $\alpha \leftrightarrow \beta$, $E \leftrightarrow D$, $\langle W| \leftrightarrow |V\rangle$ and at the same time inverting the order of the enumeration of cells (site i is replaced by site $L-i+1$). The ansatz (2.22) together with (2.40-2.51) are a notation for recursion relations for the weights on system size and particle number [24], as (2.19-2.21) for continuous time. For example (2.40) can be written as $F_L(\dots 1100\dots) = (1-p)F_{L-1}(\dots 110\dots) + (1-p)F_{L-1}(\dots 100\dots) + p(1-p)F_{L-2}(\dots 10\dots)$. On the other hand (2.51) leads to $F_L(01\dots) = p/\beta F_{L-1}(1\dots)$. To reduce these quartic equations to quadratic ones they performed the ansatz

$$E = \begin{pmatrix} E_1 & gD_1 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} D_1 & 0 \\ gE_1 & 0 \end{pmatrix}, \quad (2.52)$$

$$\langle W| = \langle\langle W_1|, \frac{\alpha}{(1-\alpha)g}\langle W_1|, \quad |V\rangle = ||V_1\rangle, \frac{\beta}{(1-\beta)g}|V_1\rangle. \quad (2.53)$$

Here the matrices E and D are effectively rank-four tensors, since its components are itself quadratic matrices. These operators fulfill the quartic algebra (2.40-2.51) if

$$D_1E_1 = (1-p)[D_1 + E_1 + p], \quad (2.54)$$

$$\langle W_1|E_1 = \frac{p(1-\alpha)}{\alpha}\langle W_1|, \quad (2.55)$$

$$D_1|V_1\rangle = \frac{p(1-\beta)}{\beta}|V_1\rangle. \quad (2.56)$$

They determine the following representation:

$$E_1 = \begin{pmatrix} p(1-\beta)/\beta & b & 0 & 0 & \dots \\ 0 & (1-p) & \sqrt{1-p} & 0 & \dots \\ 0 & 0 & (1-p) & \sqrt{1-p} & \dots \\ 0 & 0 & 0 & (1-p) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \quad (2.57)$$

$$D_1 = \begin{pmatrix} p(1-\alpha)/\alpha & 0 & 0 & 0 & \dots \\ b & (1-p) & 0 & 0 & \dots \\ 0 & \sqrt{1-p} & (1-p) & 0 & \dots \\ 0 & 0 & \sqrt{1-p} & (1-p) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \quad (2.58)$$

$$\langle W| = (1, 0, 0, \dots), \quad |V\rangle = (1, 0, 0, \dots)^t \quad (2.59)$$

where the constant b which is given through

$$b^2 = \frac{p[(1-p) - (1-\alpha)(1-\beta)]}{\alpha\beta}. \quad (2.60)$$

All physical quantities can be expressed through the indexed operators D_1, E_1 and $\langle W_1|, |V_1\rangle$. However note that the weights obtained through (2.52) have the disadvantage that they become very difficult expressions. For an arbitrary weight one obtains a complex sum over matrix elements involving indexed operators, that do not have an obvious physical meaning. For the proof of the steady state one can use the canceling-mechanism suggested for this process and update by Rajewsky [13, 63]. The idea is the following: One mimics the parallel update rule by a forward-sequential update from the left to the right. At the left end of the chain a disturbance is created that moves through the whole chain until it is destroyed at the right end. The position of the update is marked by a bared matrix. To avoid the case that a particle moves more then once, one introduced a third state \bar{F} denoting a particle that has already moved.

$$\bar{E}E = E\bar{E}, \quad (2.61)$$

$$\bar{E}D = E\bar{D}, \quad (2.62)$$

$$p\bar{D}E = E\bar{F}, \quad (2.63)$$

$$(1-p)\bar{D}E + \bar{F}E = D\bar{E}, \quad (2.64)$$

$$\bar{D}D + \bar{F}D = D\bar{D}, \quad (2.65)$$

as bulk equations and for the boundaries:

$$(1-\alpha)\langle W|E = \langle W|\bar{E}, \quad (2.66)$$

$$\langle W|D = \langle W|\bar{D}, \quad (2.67)$$

$$\alpha\langle W|E = \langle W|\bar{F}, \quad (2.68)$$

$$(\bar{E} + \beta\bar{D})|V\rangle = E|V\rangle, \quad (2.69)$$

$$((1-\beta)\bar{D} + \bar{F})|V\rangle = D|V\rangle. \quad (2.70)$$

If one can find representations for the different operators the proof of the stationary state is completed. Here we content ourselves with a special case. There is a line in the parameter space, namely $1-p = (1-\alpha)(1-\beta)$, for which the matrices E and D in (2.52) can be chosen to be two-dimensional. We find

$$\bar{E} = (1-p) \begin{pmatrix} E_1 & gD_1 \\ 0 & 0 \end{pmatrix}, \bar{D} = \begin{pmatrix} D_1 + p & 0 \\ 0 & 0 \end{pmatrix}, \bar{F} = g\alpha/p \begin{pmatrix} 0 & 0 \\ E_1^2 & gE_1D_1 \end{pmatrix}. \quad (2.71)$$

In [13] it is shown that for this choice of the parameters the state can be written as a pair-factorized state:

$$F_L(\tau_1, \tau_2, \dots, \tau_L) = h(\tau_1)t(\tau_1, \tau_2)t(\tau_2, \tau_3) \dots t(\tau_{L-1}, \tau_L)h(\tau_L). \quad (2.72)$$

Here the factors h reflect the influence of the boundaries.

In chapter 4 we will show that the steady-state weights of this process can *in general* be written as a pair-factorized state (2.72) and a matrix-product state. It is further argued that this form should be generic for a broader class of processes with parallel dynamics. An example is the process considered in chapter 5.

The bond-oriented solution

Alternatively de Gier and Nienhuis [25] performed a bond-oriented matrix ansatz:

$$P(\tau_1, \tau_2, \dots, \tau_L) = \langle W(\tau_1) | M(\tau_1, \tau_2) M(\tau_2, \tau_3) \dots M(\tau_{L-1}, \tau_L) | V(\tau_L) \rangle \quad (2.73)$$

The vectors and matrices $M_{\tau\sigma}$ are

$$M(\tau\sigma) = \begin{pmatrix} (1-\tau)(1-\sigma)\mathcal{M}(00) & (1-\tau)\sigma\mathcal{M}(01) \\ \tau(1-\sigma)\mathcal{M}(10) & \tau\sigma\mathcal{M}(11) \end{pmatrix}, \quad (2.74)$$

$$\langle W(\tau) | = ((1-\tau)\langle \mathcal{W}(0) |, \tau\langle \mathcal{W}(1) |), \quad (2.75)$$

$$|V(\tau)\rangle = ((1-\tau)|\mathcal{V}(0)\rangle, \tau|\mathcal{V}(1)\rangle)^t. \quad (2.76)$$

Through this ansatz the authors were able to reproduce the exact asymptotic behavior of the ASEP, especially the phase diagram. This is presented in the following section.

The phase diagram

The phase diagram of the ASEP with parallel update and open boundaries is shown in figure 2.3.

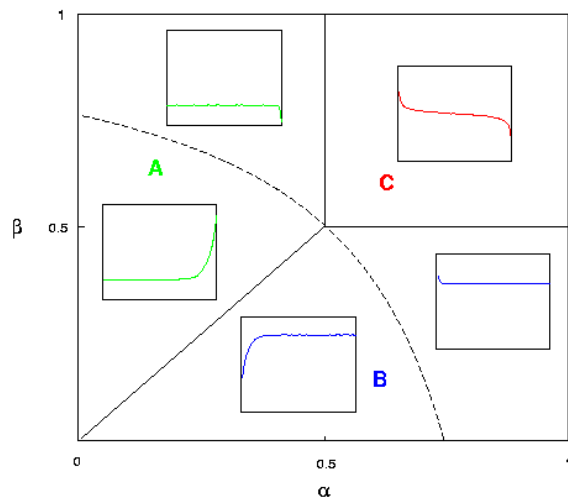


Figure 2.3: Phase diagram of the ASEP with parallel dynamics for $p = 3/4$, taken from [3].

As for random-sequential update one finds a high-density, low-density and a maximum-current phase. The dotted curve is the line $1 - p = (1 - \alpha)(1 - \beta)$ where the solution takes a simple form, as discussed above. In the windows, the density profiles are depicted schematically.

2.3 Zero-range process and mass-transport models

The Zero-Range process (ZRP) appeared first in the mathematical literature in [52]. It attracted lots of interest in the past decade, for a recent review see [19]. Zero-range processes can be defined on an arbitrary lattice and in arbitrary dimension. Since the ZRP is a model with infinite-state space, each site can be occupied by an arbitrary amount of particles. A particle on site i can hop onto another site with a certain rate that depends only on the number m_i of particles on the site of departure. This number is referred to as the ‘mass’ on site i . We are interested here in completely asymmetric hopping on a one-dimensional lattice where particles are allowed to hop only one site to the left. Note that only one particle is allowed to move at a time step. The steady state of this process is known to factorize into factors for each mass. This statement holds independently of the choice of hop rates $u(m)$.

In [64] a class of models was investigated that is referred to as mass-transport models. The paper deals with a generalization of the one-dimensional asymmetric zero-range process. Here the mass at a site can be a continuous quantity and any fraction of mass can be transferred to the neighboring site. This is also interesting if one considers simply discrete masses. Then the generalized zero-range process is defined by a chipping function $\gamma(l|m)$ denoting the rate at which l particles may leave a site that is occupied by m particles. The authors derived a condition on the chipping function γ for the steady state to factorize (as the ZRP) which reads for continuous time:

$$\gamma(l|m) = \frac{x(l)f(m-l)}{f(m)}. \quad (2.77)$$

If a relation like (2.77) with an arbitrary non-negative function $x(l)$ and single site weights $f(m)$ can be found, the invariant measure factorizes. Factorizing the weights it is easy to check that chipping functions chosen according to (2.77) fulfill the master equation [19]. They considered also a parallel update. Here we refer to the literature [64, 65]. Both the ZRP and the GZRP can exhibit condensation transitions which is the reason why it is of such a large theoretical interest. For a review on condensation in these models see [19]. For recent investigations of condensation in the zero-range process [66–68] and in mass-transport models see [69, 70].

3 The ASEP with excess-mass formation and random-sequential dynamics

Consider an ASEP with N particles on a ring of L sites. As usual for driven-diffusive systems the process is defined by its local transition rules. Particles can move one or two sites to the right. The system evolves in continuous time according to the following rules (an occupied site is denoted by 1 and an empty site by 0):

$$\begin{aligned} 100 &\rightarrow 001, & \text{with rate } 1, \\ 101 &\rightarrow 011, & \text{with rate } \beta. \end{aligned}$$

In general one might think of an additional transition $100 \rightarrow 010$. In chapter 6 this more general process which has been introduced as a simple traffic model in [71] is investigated. Here we consider the more simple process since it turns out to be a very interesting special case since it is exactly solvable and exhibits global phase transitions. In chapter 5 we also obtain the exact solution for parallel dynamics.

For the formulation of the solution it turns out to be more convenient to consider the corresponding mass-transport model. The steady state can then be characterized by the mass variables m_i denoting the number of particles present at site i . If there is a single particle present at site i it may hop with rate β to site $i - 1$. If there is more than one particle present at site i , two of these particles can hop to site $i - 1$ with rate 1. Thus we have:

$$\gamma(1|m) = \beta, \quad m = 1, \quad (3.1)$$

$$\gamma(2|m) = 1, \quad m \geq 2. \quad (3.2)$$

During the time evolution the number of odd sites can not increase due to the allowed transitions. However sites with even mass can be generated if mass m_{i-1} is an odd number and $m_i = 1$: if the single particle jumps to the left then both sites have even mass, namely $m_{i-1} + 1$ and $m_i = 0$. We find that therefore there is an intrinsic extinction of odd masses. In the steady state one finds a distinction in the form of the weights with respect to the parity of the total mass. For even total mass every stationary configuration exhibits only even masses. For odd mass stationary configurations have a single odd mass and even masses spread over all other lattice sites. All other configurations one might think of are not stationary and accordingly have weight zero. These facts will be proven below.

We find the exact steady state for arbitrary N and $M = \sum_{i=1}^N m_i$ in the following form:

- For *even mass* M we find that the steady state can be factorized:

$$P(m_1, m_2, \dots, m_N) = Z_{N, M \text{ even}}^{-1} \prod_{i=1}^N f(m_i), \quad (3.3)$$

with some single-site weights $f(m)$ for each lattice site given by

$$f(m) = \begin{cases} 1, & \text{if } m \text{ is even,} \\ 0, & \text{if } m \text{ is odd.} \end{cases} \quad (3.4)$$

- For *odd mass* M the steady-state turns out to be of matrix-product form:

$$\begin{aligned} P(m_1, m_2, \dots, m_N) &= 0, \quad \text{if more than one } m_i \text{ is odd,} \\ &= Z_{N, \text{Modd}}^{-1} \text{Tr} \left[\mathcal{U}_{m_1} \prod_{i=2}^N \mathcal{G}_{m_i} \right], \quad \text{if only } m_1 \text{ is odd.} \end{aligned} \quad (3.5)$$

In (3.5) we fixed mass 1 to be the odd one, what can always be done since the probability for any configuration is unaffected by cyclic permutation of the masses and since we only have one odd mass. For a better readability we write an operator \mathcal{G}_{2i} for even occupations and \mathcal{U}_{2i+1} for odd occupations. To ensure that (3.5) indeed gives the steady state it is sufficient that the matrices fulfill the following quadratic algebra:

$$\mathcal{G}_{2i} \mathcal{U}_1 = \mathcal{U}_{2i+1}, \quad (3.6)$$

$$\mathcal{U}_{2i+1} \mathcal{G}_{2j+2} - \mathcal{U}_{2i+1} \mathcal{G}_{2j} = 0, \quad (3.7)$$

$$\mathcal{G}_{2i} \mathcal{G}_{2j+2} - \mathcal{G}_{2i} \mathcal{G}_{2j} = \beta \mathcal{G}_{2i+2j+2}, \quad (3.8)$$

$$\mathcal{G}_{2i} \mathcal{U}_{2j+3} - \mathcal{G}_{2i} \mathcal{U}_{2j+1} = \beta \mathcal{U}_{2i+2j+3}, \quad (3.9)$$

$$(\mathcal{U}_{2i+1} \mathcal{U}_{2j+1} = 0), \quad \text{for } i, j \geq 0. \quad (3.10)$$

Note that this algebra has been guessed from solutions of finite systems. Here we do not have to consider (3.10) since we think only of configurations with only one ' \mathcal{U} '. A representation of these matrices is the following:

$$\mathcal{U}_{2i+1} = E^i A, \quad \mathcal{G}_{2i} = \beta E^i D, \quad i \geq 0. \quad (3.11)$$

The matrices E , D and A are the well-known matrices of the ASEP with a single defect particle. Performing the products and using a certain representation of E , D and A yields explicitly:

$$\mathcal{U}_{2n+1} = \begin{pmatrix} \binom{n}{0} & 0 & 0 & \dots \\ \binom{n}{1} & 0 & 0 & \dots \\ \binom{n}{2} & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \quad (3.12)$$

$$\mathcal{G}_{2n} = \begin{pmatrix} \binom{n}{0} & \binom{n+1}{0} \beta & 0 & 0 & \dots \\ \binom{n}{1} & \binom{n+1}{1} \beta & \binom{n+1}{0} \beta & 0 & \dots \\ \binom{n}{2} & \binom{n+1}{2} \beta & \binom{n+1}{1} \beta & \binom{n+1}{0} \beta & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad \text{for } n \geq 0. \quad (3.13)$$

We note that this is to our knowledge the first matrix-product solution of a mass-transport model [64] as introduced by Evans et.al. beyond scalar factorization. Also for (continuous-time) asymmetric exclusion processes with maximum velocity two only very simple limits could be solved by the matrix-product ansatz [72] or have not been worked out [73].

The fact that the steady state is factorizable for even mass but not for odd mass comes as a surprise. Beyond that it is not clear why the factorization is not predicted by the factorization criterion (2.77) (which can be checked by simply considering $\gamma(1|m)$). An explanation and some outlook for further investigations is given in section 3.9.

3.1 Proof of the steady states for even and odd particle number

In the following we prove that the steady state has the simple product form for even and the matrix-product form for odd particle number. For arbitrary particle number we perform a matrix-product ansatz of the form

$$|F_L\rangle = \text{Tr} \left(\begin{pmatrix} \mathcal{G}_0 \\ \mathcal{U}_1 \\ \mathcal{G}_2 \\ \vdots \end{pmatrix}^{\otimes L} \right), \quad (3.14)$$

Now introducing the canceling mechanism, which we do not give here explicitly for brevity, leads to the following set of equations:

$$\bar{\mathcal{G}}_0 \mathcal{G}_0 - \mathcal{G}_0 \bar{\mathcal{G}}_0 = 0, \quad (3.15)$$

$$\bar{\mathcal{G}}_0 \mathcal{G}_{2(i+1)} - \mathcal{G}_0 \bar{\mathcal{G}}_{2(i+1)} = -\mathcal{G}_0 \mathcal{G}_{2(i+1)}, \quad (3.16)$$

$$\bar{\mathcal{G}}_{2(i+1)} \mathcal{G}_0 - \mathcal{G}_{2(i+1)} \bar{\mathcal{G}}_0 = \mathcal{G}_{2i} \mathcal{G}_2 + \beta \mathcal{D}_{2i+1} \mathcal{U}_1, \quad (3.17)$$

$$\bar{\mathcal{G}}_{2(i+1)} \mathcal{G}_{2(j+1)} - \mathcal{G}_{2(i+1)} \bar{\mathcal{G}}_{2(j+1)} = \mathcal{G}_{2i} \mathcal{G}_{2(j+2)} - \mathcal{G}_{2(i+1)} \mathcal{G}_{2(j+1)}, \quad (3.18)$$

$$\bar{\mathcal{G}}_0 \mathcal{U}_1 - \mathcal{G}_0 \bar{\mathcal{U}}_1 = -\beta \mathcal{G}_0 \mathcal{U}_1, \quad (3.19)$$

$$\bar{\mathcal{G}}_0 \mathcal{U}_{2i+3} - \mathcal{G}_0 \bar{\mathcal{U}}_{2i+3} = -\mathcal{G}_0 \mathcal{U}_{2i+3}, \quad (3.20)$$

$$\bar{\mathcal{G}}_{2(i+1)} \mathcal{U}_1 - \mathcal{G}_{2(i+1)} \bar{\mathcal{U}}_1 = -\beta \mathcal{G}_{2(i+1)} \mathcal{U}_1 + \mathcal{G}_{2i} \mathcal{U}_3, \quad (3.21)$$

$$\bar{\mathcal{G}}_{2(i+1)} \mathcal{U}_{2j+3} - \mathcal{G}_{2(i+1)} \bar{\mathcal{U}}_{2j+3} = \mathcal{G}_{2i} \mathcal{U}_{2j+5} - \mathcal{G}_{2(i+1)} \mathcal{U}_{2j+3}, \quad (3.22)$$

$$\bar{\mathcal{U}}_1 \mathcal{G}_0 - \mathcal{U}_1 \bar{\mathcal{G}}_0 = \beta \mathcal{G}_0 \mathcal{U}_1, \quad (3.23)$$

$$\bar{\mathcal{U}}_1 \mathcal{G}_{2(i+1)} - \mathcal{U}_1 \bar{\mathcal{G}}_{2(i+1)} = -\mathcal{U}_1 \mathcal{G}_{2(i+1)}, \quad (3.24)$$

$$\bar{\mathcal{U}}_{2i+3} \mathcal{G}_0 - \mathcal{U}_{2i+3} \bar{\mathcal{G}}_0 = \mathcal{U}_{2i+1} \mathcal{G}_2 + \beta \mathcal{G}_{2(i+1)} \mathcal{U}_1, \quad (3.25)$$

$$\bar{\mathcal{U}}_{2i+3} \mathcal{G}_{2(i+1)} - \mathcal{U}_{2i+3} \bar{\mathcal{G}}_{2(i+1)} = \mathcal{U}_{2i+1} \mathcal{G}_{2(i+2)} - \mathcal{U}_{2i+3} \mathcal{G}_{2(i+1)}, \quad (3.26)$$

$$(\bar{\mathcal{U}}_1 \mathcal{U}_1 - \mathcal{U}_1 \bar{\mathcal{U}}_1 = -\beta \mathcal{U}_1 \mathcal{U}_1, \quad (3.27)$$

$$(\bar{\mathcal{U}}_1 \mathcal{U}_1 - \mathcal{U}_1 \bar{\mathcal{U}}_1 = -\beta \mathcal{U}_1 \mathcal{U}_1, \quad (3.28)$$

$$\bar{\mathcal{U}}_1 \mathcal{U}_{2i+3} - \mathcal{U}_1 \bar{\mathcal{U}}_{2i+3} = -\mathcal{U}_1 \mathcal{U}_{2i+3}, \quad (3.29)$$

$$\bar{\mathcal{U}}_{2i+3} \mathcal{U}_1 - \mathcal{U}_{2i+3} \bar{\mathcal{U}}_1 = -\beta \mathcal{U}_{2i+3} \mathcal{U}_1 + \mathcal{U}_{2i+1} \mathcal{U}_3, \quad (3.30)$$

$$\bar{\mathcal{U}}_{2i+3} \mathcal{U}_{2j+3} - \mathcal{U}_{2i+3} \bar{\mathcal{U}}_{2j+3} = -\mathcal{U}_{2i+3} \mathcal{U}_{2j+3} + \mathcal{U}_{2i+1} \mathcal{U}_{2j+5}. \quad (3.31)$$

From the structure of these canceling equations the following ansatz for the tagged matrices has been guessed:

$$\bar{\mathcal{G}}_0 = \mathcal{G}_0 - \beta \mathbb{1}, \quad (3.32)$$

$$\bar{\mathcal{G}}_{2(i+1)} = \mathcal{G}_{2(i+1)} + \mathcal{G}_{2i}, \quad (3.33)$$

$$\bar{\mathcal{U}}_1 = \mathcal{U}_1, \quad (3.34)$$

$$\bar{\mathcal{U}}_{2i+3} = \mathcal{U}_{2i+3} + \mathcal{U}_{2i+1}. \quad (3.35)$$

Indeed one can check that this reduces the algebra (3.15-3.31) to (3.6-3.10).

For even particle number a (one-dimensional) solution of the algebra (3.6-3.10) is simply $\mathcal{U}_{2i+1} = f(2i+1) = 0$ and $\mathcal{G}_{2i} = f(2i) = 1$.

For odd particle number the ansatz (3.11) gives

$$DE = D + E, \quad (3.36)$$

$$AE = A, \quad (3.37)$$

$$\beta DA = A, \quad (3.38)$$

$$(AA = 0). \quad (3.39)$$

Again the last relation $AA = 0$ is redundant since we have only one A and the algebra reduces to the algebra of the ASEP with a single defect particle. These results can be used to obtain the solution of a ZRP with 2 species of particles, see section 7.1.

3.2 Interpretation of the algebra (3.6)-(3.10)

The relations enable in principle to calculate a weight for system size $N + 1$ and particle number $M + 2$ recursively in the following schematic way:

$$\text{Weight}(N + 1, M + 2) = \text{Weights}(N + 1, M) + \text{Weights}(N, M + 2). \quad (3.40)$$

For simplicity we now write \mathcal{E} for \mathcal{G}_0 . For \mathcal{G}_{2i} with $i \geq 1$ but without spending attention to the value of the special index $2i$ we simply write \mathcal{G} and for \mathcal{U}_{2i+1} with $i \geq 1$ we write \mathcal{U} . At first we repeat that applying the matrix relations (3.6)-(3.10) to any word with more than one \mathcal{U} can be reduced to zero. Therefore we consider only configurations with exactly one \mathcal{U} .

Using the relations (3.6) and (3.7) any configuration (or word to be more precisely) can for $N \geq 2$ be brought into the form

$$\mathcal{U} \underbrace{\mathcal{E} \dots \mathcal{E}}_{\geq 1} \underbrace{\mathcal{G} \dots \mathcal{G}}_{\geq 0} \underbrace{\mathcal{E} \dots \mathcal{E}}_{\geq 0} \dots \underbrace{\mathcal{G} \dots \mathcal{G}}_{\geq 0} \underbrace{\mathcal{E} \dots \mathcal{E}}_{\geq 0}. \quad (3.41)$$

The subscripts ≥ 0 and ≥ 1 denote that each block may at least have length 0 or 1 respectively. What we have done is the following: If there is a \mathcal{U}_1 in the word one begins with this \mathcal{U}_1 which can be transported to the left until it reaches a \mathcal{G} . All \mathcal{E} s on its way can be omitted according to (3.6). Then the \mathcal{U}_1 and the \mathcal{G}_{2i+2} merge and form a \mathcal{U}_{2i+3} what is displayed here as \mathcal{U} . The second thing is: If there is a \mathcal{G} to the right of the \mathcal{U} in the starting configuration it can be replaced by an \mathcal{E} according to (3.7).

In other words if the rules (3.6-3.7) can be applied to a configuration then the weight for the configuration can already be found in the solution space of smaller N and/or M .

For all other configurations (of the form (3.41)) to which rules (3.6-3.7) can not be applied the corresponding weight occurs for the first time.

3.3 The question of a unified representation

The matrix or tensor ansatz (3.14) which we repeat here:

$$|F_L\rangle = \text{Tr} \left(\begin{pmatrix} \mathcal{G}_0 \\ \mathcal{U}_1 \\ \mathcal{G}_2 \\ \vdots \end{pmatrix} \right)^{\otimes L} \quad (3.42)$$

holds independently of the parity of mass. It contains the solution for even mass (where the tensors become numbers) and the solution for odd mass (including the vanishing

probabilities for configurations with more than one odd mass). Thus it would be nice if one could write a general matrix representation that contains all three special cases. However we will proof that under the ansatz (3.42) this is impossible. This can be seen as follows:

The matrix relation (3.10) implies that $(\mathcal{U}_{2i+1})^2 = 0$ for all i . So for a general representation (including the results for even and odd particle number) the operators \mathcal{U}_{2i+1} had to be non-vanishing nilpotent matrices. However for this equation to hold all eigenvalues of \mathcal{U}_{2i+1} have to be equal to zero. Therefore also $\text{tr } \mathcal{U}_{2i+1} = 0$ which implies for example through relation (3.6) that also $\text{tr } \mathcal{G}_{2i}\mathcal{U}_1 = 0$ which is untrue. Now this argument can be used successively to see that for each particle-number probabilities for configurations incorrectly give zero due to vanishing traces. Therefore the cases even/odd have to be treated separately.

We give two examples that follow directly from the algebra (3.6-3.9). Using (3.9) iteratively one gets

$$\mathcal{E}^K \mathcal{U}_{2J+1} = \mathcal{U}_1 \sum_{i=1}^K \binom{J+i-2}{J-1} \beta^{i-1} + \beta^K \sum_{i=1}^J \mathcal{U}_{2i+1} \binom{K+J-i-1}{K-1}, \quad J, K \geq 1. \quad (3.43)$$

From this relation it is now rather obvious that any word corresponding to configurations with only one occupied site has trace zero (since $\mathcal{U}_1, \mathcal{U}_3, \dots$ have trace zero) which is obviously not true.

We further found a relation for products $\mathcal{G} \dots \mathcal{G}\mathcal{U}$. They can be expressed as a linear combination of \mathcal{U} matrices:

$$\begin{aligned} \mathcal{G}_{2J_n} \dots \mathcal{G}_{2J_1} \mathcal{U}_{2J_0+1} &= a_n \mathcal{U}_{2J_0+1} \\ &+ \sum_{k=1}^n \beta^k a_{n-k} \sum_{i_1=1}^{J_{n-k}} \sum_{i_2=1}^{2(J_{n-k+1}+i_1)} \dots \sum_{i_k=1}^{2(J_{n-1}+i_1+\dots+i_{k-1})} \mathcal{U}_{2(J_n+i_1+\dots+i_k)+1}, \end{aligned} \quad (3.44)$$

where the coefficients a_n are recursively given by

$$a_0 = 1, \quad (3.45)$$

$$a_1 = 1, \quad (3.46)$$

$$a_n = a_{n-1} + \sum_{k=1}^{n-1} \beta^k a_{n-k-1} \sum_{i_1=1}^{J_{n-k-1}} \sum_{i_2=1}^{2(J_{n-k}+i_1)} \dots \sum_{i_k=1}^{2(J_{n-2}+i_1+\dots+i_{k-1})} 1. \quad (3.47)$$

In a unified representation also the trace over these words would incorrectly give zero. In section 3.5 we will see how by a slight change of the ansatz the problem can be solved. The way out the closest to our calculations here is to generalize the trace operation. Beyond that also two other possibilities.

3.4 Connection with the Defect ASEP

The solution (3.11) indeed suggests a connection with defect-ASEP. However it turns out that this is not an obvious simple mapping:

Think again in the picture of the exclusion process. Then each site is either occupied by exactly one particle or is empty. A configuration can be written as a string of 1s and 0s, for example like 1001000011001000001... If there is only one free site in front of a particle it can move this single site with rate β .

The trick now is to read off each block of 0s in the string from the left to the right. The first pair 00 in each of these blocks transform into a single 0. If to the right of this pair 00 there is another pair 00 transform it also into a single 0, and so on. But if one arrives at a single 0, followed by a 1, i.e. an 01 pair then transform this pair into a 2. Of course this happens if the 0 block consisted of an odd number of 0s.

Doing so with the string displayed above as an example, one gets

$$1001000011001000001 \dots \longrightarrow 101001101002 \dots \quad (3.48)$$

As explained in previous section there exists exactly one odd valued gap and therefore we have after the transformation exactly one 2 in the system.

Consider now the possible transitions that may happen in the presence of this single ‘defect’ particle:

- $(1\underbrace{0\dots0}_x)100 \longrightarrow (1\underbrace{0\dots0}_x)001$, at rate 1.

If $x = 0, 2, 4, \dots$ then this transition becomes $10 \longrightarrow 01$ at rate 1.

If $x = 1, 3, 5, \dots$ then this transition becomes $20 \longrightarrow 02$ at rate 1.

- $(1\underbrace{0\dots0}_x)101 \longrightarrow (1\underbrace{0\dots0}_x)011$, at rate β ,

where $x = 0, 2, 4, \dots$. This transition becomes $12 \longrightarrow 21$ at rate β .

When the steady state is not yet reached it may also appear that $x = 1, 3, 5, \dots$. This gives transitions like $22 \longrightarrow 011$.

In the language of the two-species ASEP this can be summarized as

$$\begin{aligned} 10 &\longrightarrow 01, & \text{at rate } 1, \\ 20 &\longrightarrow 02, & \text{at rate } \alpha = 1, \\ 12 &\longrightarrow 21, & \text{at rate } \beta. \end{aligned}$$

Now one can use the matrices for the two-species process, i.e. an E for a 0, a D for a 1 and an A for the 2-particle.

In our original process each configuration in the steady state can then be expressed as follows. Take the right end of the odd-valued 0 block in the string of the configuration. Represent the 01 pair by an A matrix. Then represent each 00 pair by an E and each 1 by a D . Then taking the trace over the matrix product gives the correct weight.

Finally we mention that it would be more natural to have a matrix product in the traffic picture with one matrix for each site. Hence one would search for a matrix e with $e^2 = E$ and $eD = A$. However it is not possible to construct such a matrix. This can for example be seen by formally inverting D to obtain $e = AD^{-1}$. Then one directly sees that $(AD^{-1})^2 = E$ can never be fulfilled.

3.5 Unified formulation of the exact solution by change of ansatz

It has been shown that under the usual matrix ansatz the steady state can not be written in a unified way for even and odd number of holes. Note that general problems of representation for periodic boundary conditions have been observed previously [74]. These

are in contrast to open boundary conditions where one usually can find a matrix-product state in principle [75] However we will show now three different possibilities how the state can be expressed in a unified way by a slight change of the ansatz.

3.5.1 Modification of the trace operation

If the trace operation (sum of the traces of the matrices in the diagonal elements) is changed then the algebra has a general representation. Take as ansatz

$$|P_L\rangle \propto \tilde{\text{Tr}} \left(\begin{pmatrix} \mathcal{G}_0 \\ \mathcal{U}_1 \\ \mathcal{G}_2 \\ \vdots \end{pmatrix} \right)^{\otimes L}. \quad (3.49)$$

Setting again $\mathcal{U}_{2i+1} = \mathcal{E}^i \mathcal{A}$ and $\mathcal{G}_{2i} = \beta \mathcal{E}^i \mathcal{D}$ with new tensors \mathcal{E} and \mathcal{D} reduces to the algebra $\mathcal{D}\mathcal{E} = \mathcal{D} + \mathcal{E}$, $\mathcal{A}\mathcal{E} = \mathcal{A}$, $\beta\mathcal{D}\mathcal{A} = \mathcal{A}$, and $\mathcal{A}^2 = 0$. Now one can take as a representation two by two matrices with elements built by the known E, D, A matrices of the DEHP algebra with the help of the following two by two matrices:

$$\mathbb{1} = |1\rangle\langle 1| + |2\rangle\langle 2|, \quad \mathbb{2} = |1\rangle\langle 2|. \quad (3.50)$$

Let $\mathcal{E} = E \otimes \mathbb{1}$, $\mathcal{D} = D \otimes \mathbb{1}$, $\mathcal{A} = A \otimes \mathbb{2}$. Here the products have to be applied to each component of $\mathbb{1}$ and $\mathbb{2}$ respectively. Expanding the product of the L \mathcal{E} , \mathcal{D} and \mathcal{A} matrices can now yield different cases:

- If the product contains more than one \mathcal{A} it gives the zero matrix $\mathbb{0}$.
- If the product contains no \mathcal{A} then it can be written as (a product of E and D matrices) $\otimes \mathbb{1}$.
- In the case where the product contains exactly one \mathcal{A} a resulting matrix which has the form $A \cdot$ (product of E and D matrices) $\otimes \mathbb{2}$ appears. Since the diagonal components of this two by two matrix are zero also the trace would vanish.

So one has different possibilities how to change the trace operation. For example the following choice yields the correct answer:

$$\begin{aligned} |P_L\rangle &\propto \tilde{\text{Tr}} \left(\begin{pmatrix} \mathcal{G}_0 \\ \mathcal{U}_1 \\ \mathcal{G}_2 \\ \vdots \end{pmatrix} \right)^{\otimes L} \propto \left(\begin{pmatrix} \tilde{\text{tr}} \mathcal{G}_0^L \\ \tilde{\text{tr}} \mathcal{G}_0^{L-1} \mathcal{U}_1 \\ \tilde{\text{tr}} \mathcal{G}_0^{L-1} \mathcal{G}_2 \\ \vdots \end{pmatrix} \right) \propto \left(\begin{pmatrix} \tilde{\text{tr}} \beta^L \mathcal{D}^L \\ \tilde{\text{tr}} \beta^{L-1} \mathcal{D}^{L-1} \mathcal{A} \\ \tilde{\text{tr}} \beta^L \mathcal{D}^{L-1} \mathcal{E} \mathcal{D} \\ \vdots \end{pmatrix} \right) \propto \left(\begin{pmatrix} \tilde{\text{tr}}(\beta^L D^L \otimes \mathbb{1}) \\ \tilde{\text{tr}}(\beta^{L-1} D^{L-1} A \otimes \mathbb{2}) \\ \tilde{\text{tr}}(\beta^L D^{L-1} E D \otimes \mathbb{1}) \\ \vdots \end{pmatrix} \right) \\ &\propto \left(\begin{pmatrix} \text{tr}(\beta^L D^L) \|\mathbb{1}\| \\ \text{tr}(\beta^{L-1} D^{L-1} A) \|\mathbb{2}\| \\ \text{tr}(\beta^L D^{L-1} E D) \|\mathbb{1}\| \\ \vdots \end{pmatrix} \right), \end{aligned} \quad (3.51)$$

where we have used in the last step the matrix norm $\|M\| = \max_{i,j} m_{ij}$.

3.5.2 Method with two auxiliary matrices

In the previous subsection a mechanism was presented with which one could map the model onto the usual ASEP in the even case and onto the ASEP with a single defect particle in the odd case by starting from an arbitrary particle and reading the configuration from the left to the right.

Inspired by this we found a formulation of the exact solution in terms of the following ansatz:

$$P_N(1, \tau_2, \dots, \tau_L) = Z_{L,N}^{-1} \text{Tr} \mathcal{T}_1 \prod_{l=2}^L [\tau_l \mathcal{D} + (1 - \tau_l) \mathcal{T}_0]. \quad (3.52)$$

Here the site 1 is chosen to be occupied. Furthermore let us assume that if there are odd-valued gaps in the system one of these can be found to the left of site 1. This is important and makes the formulation of the solution easier without losing generality. Beginning with the particle on site 1, holes are represented by a matrix \mathcal{T}_0 and particles by a \mathcal{D} . The matrix \mathcal{T}_1 represents the occupied site 1.

Then this matrix product can be transformed into a product of \mathcal{E} , \mathcal{D} and \mathcal{A} matrices by applying the following algebra which should be intuitively clear from the mechanism of the previous section:

$$(\mathcal{T}_1, \mathcal{E}, \mathcal{D}) \mathcal{T}_0^2 = (\mathcal{T}_1, \mathcal{E}, \mathcal{D}) \mathcal{E}, \quad (3.53)$$

$$(\mathcal{T}_1, \mathcal{E}, \mathcal{D}) \mathcal{T}_0 \mathcal{D} = 0, \quad (3.54)$$

$$(\mathcal{E}, \mathcal{D}) \mathcal{T}_1 = (\mathcal{E}, \mathcal{D}) \mathcal{D}, \quad (3.55)$$

$$(\mathcal{E}, \mathcal{D}) \mathcal{T}_0 \mathcal{T}_1 = (\mathcal{E}, \mathcal{D}) \mathcal{A}. \quad (3.56)$$

This are 10 equations (the notation with the commas and brackets is just shorthand). The first line represents the fact that if we start reading the matrix product with \mathcal{T}_1 (or have already transformed up to an \mathcal{E} or \mathcal{D}) and there follow two \mathcal{T}_0 matrices then this two matrices stand for a hole pair that can be transformed into an \mathcal{E} . The second line equates the product of $\mathcal{T}_0 \mathcal{D}$ to the right of \mathcal{T}_1 (or \mathcal{E} or \mathcal{D}) to zero since two or more odd gaps are forbidden and the particle on site 1 is reserved to have an odd hole cluster to the left. The next two equations transform the \mathcal{T}_1 matrix into a \mathcal{D} or an \mathcal{A} depending on the parity of the gap to the left.

Note that this algebra is such that one can only handle blocks of matrices beginning with $\mathcal{T}_1, \mathcal{E}$ or \mathcal{D} but one can never start with \mathcal{T}_0 .

Of course one also has the relations:

$$\mathcal{D} \mathcal{E} = \mathcal{D} + \mathcal{E}, \quad (3.57)$$

$$\mathcal{A} \mathcal{E} = \mathcal{A}, \quad (3.58)$$

$$\beta \mathcal{D} \mathcal{A} = \mathcal{A}. \quad (3.59)$$

A representation of the operators fulfilling (3.53-3.59) can be found in terms of rank four tensors, i.e. two by two matrices containing as elements the known matrices E, D, A :

$$\mathcal{T}_0 = \begin{pmatrix} 0 & \mathbb{1} \\ E & 0 \end{pmatrix}, \quad \mathcal{T}_1 = \begin{pmatrix} D & 0 \\ A & 0 \end{pmatrix}, \quad (3.60)$$

$$\mathcal{E} = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{D} = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}. \quad (3.61)$$

It is not difficult to see that the trace operation is always well defined and that one gets always the correct probabilities. The matrix \mathcal{T}_0 fulfills $\mathcal{T}_0^{2n} = E \otimes \mathbb{1}$ and $\mathcal{T}_0^{2n+1} = E^n \otimes \mathcal{T}_0$.

If there are only even gaps $\{2n_\mu\}$ then the trace yields the right product in terms of D, E and A matrices. If there is another odd gap one has $\text{Tr} \dots \mathcal{X} \mathcal{T}_0 \mathcal{D} \dots$ (\mathcal{X} stands either for \mathcal{E} or \mathcal{D}) and such a product gives correctly 0.

3.5.3 Matrices that depend on the parity of $L - N$

A different possibility is to write (with the knowledge that all configurations with more than one odd hole fraction have zero probability):

$$P_L(2\tilde{n}_1, \dots, 2\tilde{n}_{N-1}, 2\tilde{n}_N + \delta) = Z_{L,N}^{-1} \langle W_\delta | \prod_{\mu=1}^N [D_\delta E^{\tilde{n}_\mu}] | V \rangle, \quad (3.62)$$

where $\delta = 0$ if the gap in front of particle N is even and $\delta = 1$ if the gap is odd. Since all other gaps are even-valued one has $\delta = \delta(L - N) = (L - N) \bmod 2$. The representation of the matrices that here depend on $L - N$ is:

$$D_\delta = \begin{pmatrix} \beta^{-\delta} & \beta^{-\delta} & \beta^{-\delta} & \dots \\ 0 & \delta & \delta & \dots \\ 0 & 0 & \delta & \dots \\ 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad (3.63)$$

$$\langle W_\delta | = (1, \delta, \delta, \dots), \quad |V\rangle = (1, 0, 0, \dots)^t. \quad (3.64)$$

3.6 Calculation of relevant quantities

3.6.1 The process with even number of holes

Repeat that all stationary configurations are equally likely. These are configurations in which the number of holes in front of any particle μ is an even number, say $2n_\mu$. A configuration can then be expressed by the set of these numbers:

$$P(2n_1, \dots, 2n_N) = Z_{L,N}^{-1}, \quad (3.65)$$

where the normalization factor $Z_{L,N}$ was introduced. Note that this implies that the probability for gaps $n = 0, 1, \dots$ oscillates. The factorized form of the steady state $F(\{n_\mu\}) = \prod_{\mu=1}^N f(n_\mu)$ in terms of the hole fractions (gaps) n_μ can also be expressed as a matrix state in the single-site variables τ_i (see chapter 7):

$$F(\tau_1, \tau_2, \dots, \tau_L) = \text{Tr} \prod_{l=1}^L [\tau_l d + (1 - \tau_l) e], \quad (3.66)$$

with

$$d = \begin{pmatrix} 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \quad (3.67)$$

Introduce the matrix $C = Dz + E$ with a fugacity z fixing the average particle number. The normalization $Z_{L,N}$ is then given by

$$Z_{L,N} = \text{Tr} C^L \delta_{\sum \tau_i, N} = \{z^N\} \text{Tr} C^L, \quad (3.68)$$

where $\{z^N\}(\sum a_n z^n)$ denotes the coefficient of z^N in the power series $(\sum_n z^n)$. The matrix C^L reads

$$C^L = \begin{pmatrix} \sum_k \binom{L-k}{k-1} z^{L-2k+2} & \sum_k \binom{L-k-1}{k-1} z^{L-2k+1} & \cdots \\ \sum_k \binom{L-k-1}{k-1} z^{L-2k+1} & \sum_k \binom{L-k-2}{k-1} z^{L-2k} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \quad (3.69)$$

Calculating the trace and using Pascals triangle identity several times yields the normalization

$$Z_{L,N} = \frac{L(L-M-1)!}{N!M!} \delta_{L-N,2M}. \quad (3.70)$$

where the total number of holes is given by $2M$. The asymptotic expression reads

$$Z_{L,N} \cong \frac{1}{2^{N-1}} \frac{(1+\rho)^{N+M-1}}{\rho^N (1-\rho)^M} \delta_{L-N,2M}. \quad (3.71)$$

Equation (3.70) is easily interpreted combinatorially: For the first particle to place on the lattice there are L possible ways. One can then think of distributing $N-1$ particles and M hole pairs into $N+M-1$ boxes to obtain the above expression. From that it is straightforward to calculate correlation functions as

$$\langle \tau_i \rangle = \frac{N}{L}, \quad \langle \tau_i \tau_{i+1} \dots \tau_{i+m} \rangle = \frac{N}{L} \frac{(N-1)(N-2) \dots (N-m)}{(L-M-1) \dots (L-M-m)}. \quad (3.72)$$

The flow from site i is the expectation value

$$J = \langle \tau_i (1 - \tau_{i+1}) (1 - \tau_{i+2}) \rangle + \beta \langle \tau_i (1 - \tau_{i+1}) \tau_{i+2} \rangle = \langle \tau_i \rangle - \langle \tau_i \tau_{i+1} \rangle. \quad (3.73)$$

Using the expressions for the correlation functions (3.72) this then yields asymptotically

$$J(\rho) = \frac{2\rho(1-\rho)}{1+\rho}. \quad (3.74)$$

The velocity v is related to the flow via the hydrodynamic relation $J = \rho v$. Calculating the exact form for finite system size (just by multiplying (3.73) by L/N) gives

$$v = 2 \frac{1-\rho}{1+\rho} \left(1 + \frac{2}{L+N} + \dots + \frac{2^n}{(L+N)^n} + \dots \right). \quad (3.75)$$

3.6.2 The process with odd number of holes

Repeat that here all stationary configurations are of the matrix-product form. This configurations are those in which exactly one particle has an odd number of holes in front (all other particles have an even number of holes in front):

$$P(2n_1, \dots, 2n_{N-1}, 2n_N + 1) = Z_{L,N}^{-1} \langle W | E^{n_1} D E^{n_2} \dots D E^{n_N} | V \rangle, \quad (3.76)$$

where the vectors $\langle W |$, $|V\rangle$ and matrices D , E are the well-known operators from the open-boundary problem and the defect-ASEP on a ring [4]. Similar relations hold if the odd gap is not in front of the N th particle but in front of any of the others. As for the defect ASEP the partition function can be calculated:

$$Z_{L,N} = \frac{L}{N} \binom{N+M}{N-1} \sum_{m=1}^{\infty} m \binom{N+M-1}{N-m} \left(\frac{1-\beta}{\beta} \right)^{m-1} \delta_{L-N,2M+1}. \quad (3.77)$$

From this expression correlation functions can be derived as above. It turns out that the flow is related to the normalization by

$$J = 2 \frac{N}{L-2} \frac{Z_{L-2,N}}{Z_{L,N}} + \beta \frac{L-N}{L-1} \frac{Z_{L-1,N-1}}{Z_{L,N}}. \quad (3.78)$$

We have calculated the finite size expansion for the velocity in the case $\beta = 1$ and obtained

$$v = 2 \frac{1-\rho}{1+\rho} \left(1 + \frac{5/2}{L+N} + \dots + \frac{(1+3^{n+1})/4}{(L+N)^n} + \dots \right). \quad (3.79)$$

The correction of order $1/(L+N)$ of course holds also for $\beta \neq 1$.

To summarize in both cases (even and odd number of holes) the velocity of particles is given by

$$v = 2 \frac{1-\rho}{1+\rho} + \mathcal{O} \left(\frac{1}{L+N} \right). \quad (3.80)$$

Just the special form of the correction differs for even and odd number of holes.

3.7 Finite number of particles

Let us consider as a special case only two particles and an arbitrary number of sites. If L is even the probability for odd headway is zero and for even headway simply $P(2n) = 2/L$. If L is odd the weights are of the form $f(2l+1, 2m) = 1 + l\beta$. The probability for a certain even distance is

$$P(2k) = \frac{\text{Tr } \mathcal{G}_{2k} \mathcal{U}_{L-2-2k}}{\text{Tr } (\mathcal{G}_0 \mathcal{U}_{L-2} + \mathcal{U}_1 \mathcal{G}_{L-3}) + \dots}, \quad (3.81)$$

and for an odd distance equivalently

$$P(2k+1) = \frac{\text{Tr } \mathcal{U}_{2k+1} \mathcal{G}_{L-3-2k}}{\text{Tr } (\mathcal{G}_0 \mathcal{U}_{L-2} + \mathcal{U}_1 \mathcal{G}_{L-3}) + \dots}. \quad (3.82)$$

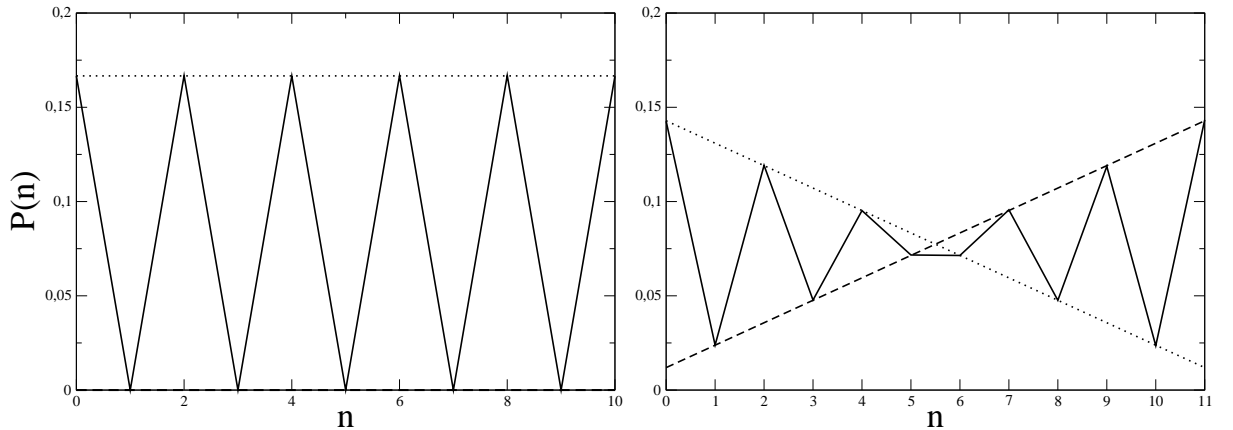


Figure 3.1: Comparison of the probabilities for certain gap sizes in the two-particle sector for $L = 12$ (left) and $L = 13$ (right) for $p_1 = 0$, $p_2 = 1$ and $\beta = 1$.

One possibility is to use the explicit representation (3.12) to obtain

$$P(2k+1) = \frac{1+k\beta}{2(1+c)(1+\frac{c}{2}\beta)}, \quad (3.83)$$

and

$$P(2k) = \frac{1 + (c - k)\beta}{2(1 + c)(1 + \frac{c}{2}\beta)}, \quad (3.84)$$

with the abbreviation $c = (L - 3)/2$. While the two-particle weights (with one odd headway and one even headway) are independent of the special size of the even headway this quantity enters in the probability for an even headway as $c - k$. For three particles we find the weight $f(2l + 1, 2m, 2j) = 1 + (l + j)\beta + \left(\frac{l(l+1)}{2} + lj\right)\beta^2$ from which the headway distribution can be calculated. The results are shown in figure 3.2a. One sees the remarkable change of the distribution by adding a single empty site to the system. In terms of a scaling function $x = 2(k + 1)/L$ which equals $(n + 1)/L$ for n odd one finds the scaling form $P(2k + 1) \sim x(1 - x)$ for odd headway. The probability for even headway scales asymptotically: $P(2k) \sim (1 - x)(1 - x + 2/L)$. As a consequence of these formulae the curve for an odd headway crosses the curve for even headway at approximately $x = 1/2$ which is astonishing. Figure 3.2b shows the case of four particles. Remember that for increasing particle number the probability $P(2k + 1)$ vanishes as N^{-1} goes to zero.

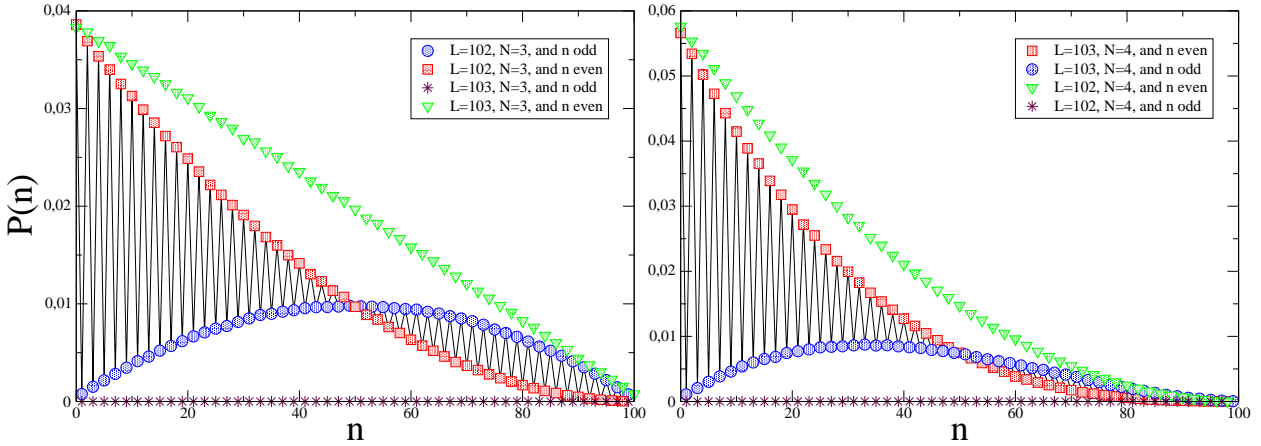


Figure 3.2: Headway distribution $P(n)$ for $p_2 = 1$, $p_1 = 0$, and $\beta = 1$ for $N = 3$ and 4 and $L = 102$ and 103 .

For the thermodynamic limit, one might directly make use of the density profile [4, 23]. We finally note that the connection between the solvability for two particles and N particles is still an open problem. Here one might be able to profit from knowledge in equilibrium statistical mechanics [76].

3.8 The phase transition

We now investigate the thermodynamic limit of the model. Since the model is related to the defect ASEP equivalently a phase transition takes place. The critical density in term of the different density as in the defect ASEP is $\rho_c = \beta/(2 - \beta)$. In the following let us in analogy refer to the 01-pair as the defect. We can carry over the results presented in the introduction. Since we have $\alpha = 1$ the phase diagram (figure 2.2) reduces to a single line crossing phases 2 and 4.

- For $\rho > \rho_c$ the defect behaves as the other particles. In front of the defect the density profile decreases exponentially to its bulk value ρ . The density behind is constant.

- For $\rho < \rho_c$ the defect is similar to a second-class particle [22] that lowers the average speed of the other particles. The density profile decays algebraically to the bulk value. Behind the defect the density is decreased and the profile increases in the same way to its bulk value as in front.

The normalization $Z_{N,M}$ (for a system with N particles and $2M + 1$ holes and therewith $L = N + 2M + 1$ sites) becomes in the thermodynamic limit in both phases

$$Z_{L,N,M} \cong \frac{1 - \rho^2}{\rho(1 - \rho/\rho_c)} \binom{M + N - 2}{N - 1}^2 \delta_{L-N,2M+1}, \text{ for } \rho < \rho_c, \quad (3.85)$$

$$Z_{L,N,M} \cong L \frac{(\rho/\rho_c - 1)}{2\rho(1 + \rho)\beta^{N-2}(1 - \beta)^M} \binom{M + N - 2}{N - 1} \delta_{L-N,2M+1}, \text{ for } \rho > \rho_c, \quad (3.86)$$

which accordingly vanish or have a pole at $\rho = \rho_c$. Compare these results with the formula in the even case by carefully expanding (3.70):

$$Z_{L,N,M} \cong \frac{1 + \rho}{\rho(1 - \rho)} \binom{M + N - 2}{N - 1} \delta_{L-N,2M}. \quad (3.87)$$

From the relation to the defect ASEP one can obtain the probabilities $P(2n + 1)$ for odd headway. For example $P(1)$ is related to the probability ρ_- in [23] to find a particle directly behind the defect. Since in our process the defect can be any of the N particles one has

$$[P(1)](\rho) = \begin{cases} \frac{4\rho^2}{\beta N(1 + \rho)^2}, & \text{for } \rho < \rho_c, \\ \frac{2\rho}{N(1 + \rho)}, & \text{for } \rho > \rho_c. \end{cases} \quad (3.88)$$

Figure 3.3 shows $P(1)$ scaled with N versus the density for $\beta = 2/3$, so that the phase transition happens at $\rho_c = 1/2$. Depicted are the analytic formulae from (3.88) together with a computer simulation for $L = 1000$ with N increased in steps of $\Delta N = 25$.

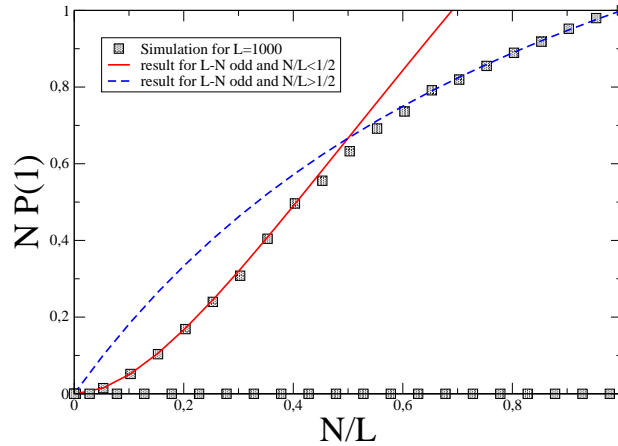


Figure 3.3: $P(1)$ versus ρ for $\beta = 2/3$. See text for details.

3.9 Product states and matrix-product states in discrete-mass transport models

Here we give a note on the fact that in the mass-transport model there exist factorized steady states that do not immediately follow from the condition (2.77).

Factorizing the master equation for the MTM and equating terms with common site index gives [19]:

$$\sum_{l=1}^n \gamma(l|m+l) f(n-l) f(m+l) = \sum_{l=1}^n \gamma(l|n) f(n) f(m). \quad (3.89)$$

The condition for this equation to hold is (2.77). However if the function $f(m)$ has any zeros, for example, then this restriction is not properly defined. To shorten notation we write for the right-hand side: $\text{rhs}(m, n) = \sum_{l=1}^n \gamma(l|n) f(n) f(m)$. We find the following relations that have to hold for all m, n and $l = 1, \dots, n$:

$$\gamma(l|m+l) = \begin{cases} \text{free,} & \text{for } f(m+l)f(n-l) = 0, \\ 0, & \text{for } f(m+l)f(n-l) > 0 \text{ and } \text{rhs}(m, n) = 0, \\ \frac{x(l)f(m)}{f(m+l)}, & \text{for } f(m+l)f(n-l) > 0 \text{ and } \text{rhs}(m, n) > 0. \end{cases} \quad (3.90)$$

The first line corresponds to the case where no restriction to γ has to be applied because the term in (3.89) in which it appears vanishes. The second line reflects the case in which the right-hand side of (3.89) is zero and because all expressions on the l.h.s. of (3.89) are non-negative, they all have to vanish identically. For all l for which $s(l)$ does not vanish therefore γ definitely has to. The last line then repeats the original condition on γ .

The condition (3.90) is now true for all N for which the weights of configurations with fixed N remain normalizable, i.e. for which the following relation holds:

$$Z_{L,N} = \sum_{\{m_i\}} \delta \left(N - \sum_j m_j \right) \prod_{i=1}^L f(m_i) > 0. \quad (3.91)$$

Assume that

$$f(m) \begin{cases} = 0, & \text{for } m \text{ odd,} \\ > 0, & \text{otherwise,} \end{cases} \quad (3.92)$$

as it is the case for (3.1–3.2). Then it turns out that

$$\gamma(l|m+l) = \begin{cases} \text{free,} & \text{for } m+l \text{ odd,} \\ 0, & \text{for } l, m \text{ both odd,} \\ \frac{x(l)f(m)}{f(m+l)}, & \text{for } l, m \text{ both even.} \end{cases} \quad (3.93)$$

Comparing with (3.90) shows that indeed this leads to a factorized state. However the choice for $f(m)$ in turn implies that the total number of particles $N = \sum_{i=1}^L m_i$ is even, since for odd particle number the normalization (3.91) would vanish. It would now be interesting to study the general model (3.93) with odd particle number defined through the single-site weights for the case of even particle number. The matrix-product solvability should hold also for more general choices of γ . Beyond that, the single-site mass-distribution should equal thermodynamically the result obtained from the case where it is factorizable, since in the infinite system a local perturbation changes the density profile but not the single-site distributions. This way one can obtain the exact distributions also for cases where the steady state has not generally a product measure. For future work it is interesting to investigate the connection between systems with creation and annihilation of ‘defects’ and generalized ZRP to be able to handle ergodic dynamics without parity dependence.

4 Alternative solution of the ASEP with open boundaries and parallel dynamics

The solutions of Evans (site-oriented) et al. [24] and de Gier (bond-oriented) [25] were presented in section 2.2.4. To map the site-oriented solution onto the bond-oriented it is useful to choose slightly different representations of the tensors E and D . This is first motivated by the solution on the ring: Equation (2.37) can be rewritten as a matrix product state

$$F_L^{\text{ring}}(\tau_1, \dots, \tau_L) = \text{tr} \prod_{l=1}^L [\tau_l D + (1 - \tau_l) E]. \quad (4.1)$$

Of course 2×2 matrices D and E of the form (2.52) solve also the process on the ring [24, 72]. In fact it has the advantage that it reflects the particle-hole symmetry of the model. However here we write the matrices in a slightly more ordinary vector basis where the $t(\tau\sigma)$ become matrix elements $\langle \tau | (E + D) | \sigma \rangle$ in the style of an Ising transfer matrix:

$$E = \begin{pmatrix} t(00) & 0 \\ t(10) & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & t(01) \\ 0 & t(11) \end{pmatrix}. \quad (4.2)$$

Here it is obvious that the product is self-consistent where terms $\dots t(\tau\sigma)t(\tau'\sigma')$ with $\sigma \neq \tau'$ do not occur. Beyond that the product leads to one single term (2.37).

Now we come to open boundaries. Inspired by (4.2) we take alternatively to (2.52)

$$E = \begin{pmatrix} t(00)E_1 & 0 \\ t(10)E_1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & t(01)D_1 \\ 0 & t(11)D_1 \end{pmatrix}. \quad (4.3)$$

Here $t(\tau\sigma)$ are the two-site factors of the solution for periodic conditions (2.39). We set $t(00) = t(10) = 1$ according to (2.39). Since the operators E or D of the form (4.3) have the structure of (4.2) with factors $t(\tau_{l-1}\tau_l)$ in a matrix for site l , the correct connection to the operator for site $l - 1$ is guaranteed. On inserting into the quartic algebra one recovers precisely (2.55, 2.56) for the first components of the boundary vectors $\langle W |$ and $| V \rangle$ which then read

$$\langle W | = \langle \langle W_1 | \frac{p - \alpha}{\alpha}, \langle W_1 |, \quad | V \rangle = | | V_1 \rangle, \frac{1 - p}{1 - \beta} | V_1 \rangle. \quad (4.4)$$

Here it was used that $\langle W |$ is defined only up to an overall factor. For the bulk one finds

$$t(01)D_1E_1 = t(11)D_1 + E_1 + p. \quad (4.5)$$

So setting

$$t(01) = (1 - p)^{-1}, \text{ and } t(11) = 1, \quad (4.6)$$

recovers (2.54). On using the new operators, the probability for a configuration $\{\tau_1, \dots, \tau_L\}$ can be written as a product of a pair-factorized (reflecting the nearest-neighbor correlations

of the parallel update) and a matrix-product state (as for other discrete-time updates such as ordered-sequential and sublattice-parallel updates [13, 24]):

$$P(\tau_1, \dots, \tau_L) = Z_L^{-1} \tilde{t}(\tau_1) t(\tau_1, \tau_2) \dots t(\tau_{L-1}, \tau_L) \tilde{t}(\tau_L) \times \langle W_1 | \prod_{l=1}^L [\tau_l D_1 + (1 - \tau_l) E_1] | V_1 \rangle \quad (4.7)$$

with $t(\tau\sigma)$ defined through (2.39) and boundary factors:

$$\tilde{t}(\tau_1) = \frac{p - \alpha}{\alpha} t(01)^{\tau_1} + t(11)^{\tau_1}, \quad \tilde{t}(\tau_L) = \frac{1 - p\tau_L}{1 - \beta\tau_L}. \quad (4.8)$$

We give an alternative representation to (2.57) that will be used later on:

$$E_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ (1-p) & 0 & 0 & 0 & \dots \\ 0 & (1-p) & 0 & \dots & \\ 0 & 0 & (1-p) & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad (4.9)$$

$$D_1 = \begin{pmatrix} p(1-\beta)/\beta & p/\beta & p/\beta & p/\beta & \dots \\ 0 & (1-p) & 1 & 1 & \dots \\ 0 & 0 & (1-p) & 1 & \dots \\ 0 & 0 & 0 & (1-p) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad (4.10)$$

$$\langle W_1 | = (1, \frac{p(1-\alpha)}{\alpha(1-p)}, \left(\frac{p(1-\alpha)}{\alpha(1-p)}\right)^2, \dots), \quad |V_1\rangle = (1, 0, 0, \dots)^t. \quad (4.11)$$

For an overview of representations for the quadratic algebra see [77]. The relation with the bond-oriented solution of de Gier and Nienhuis is as follows: In (4.3) we have to take

$$t_{01} = 1 \text{ and } t_{11} = 1 - p. \quad (4.12)$$

Then the connection is:

$$\mathcal{M}(00) = \mathcal{M}(10) = E_1, \quad (4.13)$$

$$\mathcal{M}(11) = (1-p)\mathcal{M}(01) = D_1 \quad (4.14)$$

$$\langle \mathcal{W}(0) | = \langle W_1 | E_1 + \langle W_2 | E_1, \quad (4.15)$$

$$\langle \mathcal{W}(1) | = \langle W_1 | D_1 + (1-p)\langle W_2 | D_1, \quad (4.16)$$

$$|\mathcal{V}(0)\rangle = |V_1\rangle, \quad (4.17)$$

$$|\mathcal{V}(1)\rangle = |V_2\rangle, \quad (4.18)$$

$$(4.19)$$

and therefore

$$E = M(00) + M(10), \quad (4.20)$$

$$D = M(01) + M(11), \quad (4.21)$$

$$(\langle W | E, 0) = \langle W(0) |, \quad (4.22)$$

$$(0, \langle W | D)^t = \langle W(1) |, \quad (4.23)$$

$$|V\rangle = |V(0)\rangle + |V(1)\rangle. \quad (4.24)$$

5 The ASEP with excess-mass formation and parallel dynamics

Now turn to parallel dynamics. Thus one has transition probabilities rather than rates:

$$\begin{aligned} 100 &\rightarrow 001, & \text{with probability } p, \\ 101 &\rightarrow 011, & \text{with probability } \beta. \end{aligned}$$

This process is a special case of the traffic model [71].

5.1 Exact solution

As in the case of random-sequential dynamics one finds a different form of the solution for even and odd number of holes. For even number of holes the system arranges such that there remain only even valued gaps. The weight for a configuration factorizes into N factors, one for each gap. All positive even gaps have the same weight. Only the weight for zero gap is different:

$$F(n_1, n_2, \dots, n_N) = \prod_{\mu=1}^N f(n_\mu), \quad (5.1)$$

with

$$f(n) = \begin{cases} 1 - p, & \text{for } n = 0, \\ 1, & \text{for } n = 2, 4, \dots, \\ 0, & \text{for } n = 1, 3, \dots \end{cases} \quad (5.2)$$

For odd number of holes there remains – as in the case of random-sequential dynamics – only one odd-valued gap (Configurations with more than one odd-valued gap have probability zero in the steady state). Inspired by the solution of the random-sequential model we make the following ansatz:

$$F(2n_1, 2n_2, \dots, 2n_N + 1) = \text{Tr} \left[\prod_{\mu=1}^{N-1} e^{n_\mu} \beta d \right] e^{n_N} a, \quad (5.3)$$

i.e. a factorization in operators as in the solution for random-sequential dynamics, i.e. a matrix a for the single pair 01, a matrix e for pairs 00 and a matrix D for single particles 1. From diagonalizations of small systems we found a recursive algebra for the operators e, d, a relating the weights for different system sizes. There are quartic and cubic rules. The bulk rules are:

$$ddee = dde + (1 - p)dee + pde, \quad (5.4)$$

$$(1 - p)dded = ddd + (1 - p)^2 ded + pdd, \quad (5.5)$$

$$(1 - p)edee = (1 - p)ede + (1 - p)eee + pee, \quad (5.6)$$

$$(1 - p)eded = edd + (1 - p)eed + ped. \quad (5.7)$$

The quartic boundary rules are:

$$adee = ade + aee, \quad (5.8)$$

$$(1-p)aded = add + (1-p)^2aed + pad, \quad (5.9)$$

$$(1-p)ddea = (1-p)^2dea + dda, \quad (5.10)$$

$$(1-p)edea = eda + (1-p)eea. \quad (5.11)$$

Cubic boundary relations:

$$aee = ae, \quad (5.12)$$

$$(1-p)aed = ad + pa, \quad (5.13)$$

$$\beta dda = p(1-\beta)da, \quad (5.14)$$

$$\beta eda = pea. \quad (5.15)$$

One sees that there must be a relationship to the algebra of the open-boundary ASEP. This relationship becomes obvious if we take $e \rightarrow (1-p)^{-1}E$ and $a = |V\rangle\langle W|$. Then the ansatz rereads

$$F(2n_1, 2n_2, \dots, 2n_N + 1) = \langle W | \left[\prod_{\mu=1}^{N-1} ((1-p)^{-1}E)^{n_\mu} \beta D \right] ((1-p)^{-1}E)^{n_N} | V \rangle. \quad (5.16)$$

To make the relation to the ASEP with open boundaries and parallel dynamics even more obvious we introduce a parameter $\alpha = p$ and rewrite the relations (5.8-5.11) with the help of (5.12-5.15). Then one recovers essentially (2.40-2.50) with $\alpha = p$ and just (2.51) has to be changed to $\langle W | ED = p/\alpha \langle W | D + p \langle W |$ which reads here

$$\langle W | ED = \langle W | (D + p), \quad (5.17)$$

This change is in accordance with the fact that in our model the particle-hole symmetry is broken. To calculate $\langle W | ED | V \rangle$ one can either use (5.17) or (2.47). The resulting equation $\langle W | E | V \rangle = \beta/p \langle W | D | V \rangle + \langle W | V \rangle$ can be used to determine $\langle W | E | V \rangle$ and $\langle W | D | V \rangle$. Both should be proportional to $\langle W | V \rangle$, so we take

$$\langle W | E | V \rangle = (\gamma + \beta) \langle W | V \rangle, \quad (5.18)$$

$$\langle W | D | V \rangle = p\gamma/\beta \langle W | V \rangle, \quad (5.19)$$

for some constant $\gamma > 0$. To calculate $\langle W | DE | V \rangle$ one can use either (2.47) or (2.49) leading to

$$\langle W | DE | V \rangle = (1-\beta) \langle W | D | V \rangle + (1-p) \langle W | E | V \rangle + p\gamma \langle W | V \rangle. \quad (5.20)$$

We take again $t(00) = t(10) = t(11) = 1$ and $t(01) = (1-p)^{-1}$. Then the weights can again be written as a superposition of a pair-factorized and a matrix state as (4.7). We can rewrite (5.16) as

$$F(2n_1, 2n_2, \dots, 2n_N + 1) = \langle W_1 | \left[\prod_{\mu=1}^{N-1} E_1^{n_\mu} (1 - p\theta(n_\mu))^{-1} D_1 \right] E_1^{n_N} | V_1 \rangle \frac{1 - p\delta(n_N, 0)}{1 - \beta\delta(n_N, 0)}. \quad (5.21)$$

Note that in fact $t_{01} = (1-p)^{-1}$ are three-site dependent factors, however hole-pairs are treated as units, therefore the terminology pair-factorized. In our opinion this form of the

weights helps to understand the solution of this type of models. However one can also work directly with the matrices (4.3) which read here

$$E = \begin{pmatrix} E_1 & 0 \\ E_1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & (1-p)^{-1}D_1 \\ 0 & D_1 \end{pmatrix}, \quad (5.22)$$

as well as

$$\langle W| = (0, \langle W_1|), \quad |V\rangle = \left(|V_1\rangle, \frac{1-p}{1-\beta}|V_1\rangle \right)^t. \quad (5.23)$$

This leads to the ternary algebra for the indexed matrices:

$$D_1 E_1 = (1-p)[D_1 + E_1 + p], \quad (5.24)$$

$$E_1 D_1 |V_1\rangle = \frac{p(1-\beta)}{\beta} E_1 |V_1\rangle, \quad (5.25)$$

$$D_1 D_1 |V_1\rangle = \frac{p(1-\beta)}{\beta} D_1 |V_1\rangle, \quad (5.26)$$

$$\langle W_1| E_1 E_1 = (1-p)\langle W_1| E_1, \quad (5.27)$$

$$\langle W_1| E_1 D_1 = (1-p)\langle W_1| D_1 + p(1-p)\langle W_1|. \quad (5.28)$$

Translate (5.18) into the form with indexed matrices:

$$\langle W_1| E_1 |V_1\rangle = \frac{1-p}{1-\beta}(\gamma + \beta)\langle W_1| V_1\rangle, \quad (5.29)$$

$$\langle W_1| D_1 |V_1\rangle = p\gamma/\beta\langle W_1| V_1\rangle. \quad (5.30)$$

A useful choice for γ (which coincides with the representation that we propose in the following section) is $\gamma = 1 - \beta$. Then one has:

$$\langle W_1| E_1 |V_1\rangle = \frac{1-p}{1-\beta}\langle W_1| V_1\rangle, \quad (5.31)$$

$$\langle W_1| D_1 |V_1\rangle = \frac{p(1-\beta)}{\beta}\langle W_1| V_1\rangle. \quad (5.32)$$

Now one sees why the choice $\gamma = 1 - \beta$ is so useful, namely because the algebra (5.24) simplifies to

$$D_1 E_1 = (1-p)[D_1 + E_1 + p], \quad (5.33)$$

$$D_1 |V_1\rangle = \frac{p(1-\beta)}{\beta}|V_1\rangle, \quad (5.34)$$

$$\langle W_1| E_1 E_1 = (1-p)\langle W_1| E_1, \quad (5.35)$$

$$\langle W_1| E_1 D_1 = (1-p)\langle W_1| D_1 + p(1-p)\langle W_1|, \quad (5.36)$$

since here the first two rules are quadratic.

5.1.1 Representation of the algebra

A representation can be found by changing slightly the alternative representation (4.9) that we have found the parallel ASEP:

$$E_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ (1-p) & 0 & 0 & 0 & \dots \\ 0 & (1-p) & 0 & \dots & \\ 0 & 0 & (1-p) & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad (5.37)$$

$$D_1 = \begin{pmatrix} p(1-\beta)/\beta & p/\beta & p/\beta & p/\beta & \dots \\ 0 & (1-p) & 1 & 1 & \dots \\ 0 & 0 & (1-p) & 1 & \dots \\ 0 & 0 & 0 & (1-p) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad (5.38)$$

$$\langle W_1 | = (1-\beta, 1, 1, 1, \dots), \quad |V_1\rangle = (1, 0, 0, \dots)^t \quad (5.39)$$

5.2 Proof of the steady state

There are different ways of proving a matrix-product state. The proof using the canceling mechanism would lie in finding several auxiliary matrices that fulfil the bulk algebra (2.61-2.65) and some additional relations for the region around the defect. However the parallel update on a ring is more difficult to handle as the case of an open lattice. The resulting equations are not appropriate for a use here.

Therefore we choose here a different kind of attack, namely by proving directly the quadratic and cubic rules for the matrices D_1 , E_1 and A_1 instead of the quartic rules for the matrices D , E and A .

To do this we will derive the master equation from the local dynamical rules.

5.2.1 Derivation of the master equation

We write the state of the system as the ket-vector $|n_1, n_2, \dots, n_N\rangle$, denoting particle 1 followed by n_1 holes and so on. This may formally be obtained by the tensor product of the single-particle states $|n_\mu\rangle$. Let $d_{jk}(n_\mu)$ be the transition probability for particle μ to go from state $|n_\mu + j + k\rangle$ into $|n_\mu\rangle$ on moving j cells while particle $\mu + 1$ moves k cells. Then the master equation can be written as

$$\langle F | \{n_\mu\} \rangle = \langle F | \text{tr} \prod_{\mu=1}^N T(n_\mu), \quad (5.40)$$

with the transfer matrix

$$T(n_\mu) = \begin{pmatrix} d_{00}(n_\mu)|n_\mu\rangle & d_{01}(n_\mu)|n_\mu - 1\rangle & d_{02}(n_\mu)|n_\mu - 2\rangle \\ d_{10}(n_\mu)|n_\mu + 1\rangle & d_{11}(n_\mu)|n_\mu\rangle & d_{12}(n_\mu)|n_\mu - 1\rangle \\ d_{20}(n_\mu)|n_\mu + 2\rangle & d_{21}(n_\mu)|n_\mu + 1\rangle & d_{22}(n_\mu)|n_\mu\rangle \end{pmatrix}. \quad (5.41)$$

The transition probabilities are

$$d_{0k}(n) = \delta_{k,n} + (1-\beta)\delta_{k,n-1} + (1-p)\theta(n-1-k), \quad (5.42)$$

$$d_{1k}(n) = \beta\delta_{k,n}, \quad (5.43)$$

$$d_{2k}(n) = p\theta(n-k+1), \quad (5.44)$$

what is directly obtained from the model definition.

Since we know that in the steady state there remains only one odd gap between the particles we use this to simplify the equation. Let the odd gap be between particle N and particle 1. Then we ask for the probability flow into the state $|2n_1, 2n_2, \dots, 2n_{N-1}, 2n_N + 1\rangle$. To obtain this state either particle N or particle 1 have been in the odd state before, since the odd gap can move only backwards. All other particles have been in an even state. Using this one finds for $T(2n_1)$:

$$T(2n_1) = \begin{pmatrix} (\delta_{n_1,0} + (1-p)\theta(n_1))|2n_1\rangle & 0 & (\delta_{n_1,1} + (1-p)\theta(n_1-1))|2n_1-2\rangle \\ \beta\delta_{n_1,0}|1\rangle & 0 & \beta\delta_{n_1,1}|1\rangle \\ p|2n_1+2\rangle & 0 & p\theta(n_1)|2n_1\rangle \end{pmatrix} \quad (5.45)$$

The second column vanishes because particle 2 can not have moved one site since it had an even gap in front as claimed before. Now using the matrix ansatz this can be written as

$$T(2n_1) = \begin{pmatrix} E_1^{n_1} D_1 & 0 & \theta(n_1) E_1^{n_1-1} D_1 \\ \beta \frac{1-p}{1-\beta} \delta_{n_1,0} A_1 & 0 & \beta \delta_{n_1,1} \frac{1-p}{1-\beta} A_1 \\ \frac{p}{1-p} E_1^{n_1+1} D_1 & 0 & \frac{p}{1-p} \theta(n_1) E_1^{n_1} D_1 \end{pmatrix}. \quad (5.46)$$

Equivalently one has for $\mu = 2..N-1$:

$$T(2n_\mu) = \begin{pmatrix} E_1^{n_\mu} D_1 & 0 & \theta(n_\mu) E_1^{n_\mu-1} D_1 \\ 0 & 0 & 0 \\ \frac{p}{1-p} E_1^{n_\mu+1} D_1 & 0 & \frac{p}{1-p} \theta(n_\mu) E_1^{n_\mu} D_1 \end{pmatrix}, \quad (5.47)$$

and for $T(2n_N + 1)$:

$$T(2n_N + 1) = \begin{pmatrix} (1-p) E_1^{n_N} A_1 & E_1^{n_N} D_1 & \theta(n_N) (1-p) E_1^{n_N-1} A_1 \\ 0 & 0 & 0 \\ p E_1^{n_N+1} A_1 & \frac{p}{1-p} E_1^{n_N+1} D_1 & p \theta(n_N) E_1^{n_N} A_1 \end{pmatrix}. \quad (5.48)$$

Note that the component of the second row and second column, containing a factor d_{11} , vanishes in every transfer matrix since it is impossible that a particle and its particle in front move at the same time only a single site in the steady state.

Now on inserting this matrices into the master equation one ends up with a product of bulk transfer matrices that is difficult to handle. The crucial step for the proof on the level of deriving the master equation is that we make the following similarity transform: Take

$$L = \begin{pmatrix} (1-p)E_1 & 0 & (1-p) \\ 0 & 0 & 0 \\ -pE_1 & 0 & (1-p) \end{pmatrix} \text{ and } R = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ \frac{p}{1-p}E_1 & 0 & E_1 \end{pmatrix}. \quad (5.49)$$

Then one has

$$LT(2n_\mu)R = \begin{pmatrix} E_1^{n_\mu+1} D_1 + \frac{p}{1-p} \theta(n_\mu) E_1^{n_\mu} D_1 E_1 & 0 & -E_1^{n_\mu+1} D_1 + \theta(n_\mu) E_1^{n_\mu} D_1 E_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (5.50)$$

As can be checked by straightforward calculation the product of bulk transfer matrices is

then

$$\begin{aligned} \prod_{\mu=2}^{N-1} T(2n_\mu) = R \otimes \prod_{\mu=2}^{N-2} \left(E_1^{n_\mu} D_1 + \frac{p}{1-p} \theta(n_\mu) E_1^{n_\mu-1} D_1 E_1 \right) \\ \otimes \begin{pmatrix} E_1^{n_{N-1}} D_1 & 0 & \theta(n_{N-1}) E_1^{n_{N-1}-1} D_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (5.51)$$

Now inserting the matrix ansatz on the left-hand side of the master equation and considering the whole product $T(2n_1) \prod_{\mu=2}^{N-1} T(2n_\mu) T(2n_N + 1)$ on inserting the expressions for the transfer matrices around the odd gap on the right-hand side and finally taking the trace over the resulting three by three matrix yields

$$\begin{aligned} & \frac{1}{1-p\theta(n_1)} \cdots \frac{1}{1-p\theta(n_{N-1})} \frac{1-p\delta_{n_N,0}}{1-\beta\delta_{n_N,0}} \text{tr} A_1 E_1^{n_1} D_1 \dots E_1^{n_{N-1}} D_1 E_1^{n_N} = \dots \\ & = \text{tr} \left[\left((1-p) E_1^{n_{N-1}} D_1 E_1^{n_N} A_1 + p\theta(n_{N-1}) E_1^{n_{N-1}-1} D_1 E_1^{n_N+1} A_1 \right) \right. \\ & \times \left[E_1^{n_1} D_1 + \frac{p}{1-p} \theta(n_1) E_1^{n_1-1} D_1 E_1 \right] \\ & + \left[E_1^{n_{N-1}} D_1 E_1^{n_N} D_1 + \frac{p}{1-p} \theta(n_{N-1}) E_1^{n_{N-1}-1} D_1 E_1^{n_N+1} D_1 \right] \\ & \times \left[\frac{\beta(1-p)}{1-\beta} \delta_{n_1,0} A_1 + \frac{p\beta}{1-\beta} \delta_{n_1,1} A_1 E_1 \right] \\ & + \left[p\theta(n_N) E_1^{n_{N-1}} D_1 E_1^{n_N-1} A_1 + \frac{p^2}{1-p} \theta(n_{N-1}) \theta(n_N) E_1^{n_{N-1}-1} D_1 E_1^{n_N} A_1 \right] \\ & \times \left[E_1^{n_1+1} D_1 + \frac{p}{(1-p)} \theta(n_1) E_1^{n_1} D_1 E_1 \right] \\ & \times \left. \prod_{\mu=2}^{N-2} \left(E_1^{n_\mu} D_1 + \frac{p}{1-p} \theta(n_\mu) E_1^{n_\mu-1} D_1 E_1 \right) \right] \end{aligned} \quad (5.52)$$

Now writing every trace starting with A_1 and simplifying the results yields

$$\begin{aligned} & \text{tr} A_1 \prod_{\mu=1}^{N-1} \left(\frac{1}{1-p\theta(n_\mu)} E_1^{n_\mu} D_1 \right) \frac{1-p\delta_{n_N,0}}{1-\beta\delta_{n_N,0}} E_1^{n_N} \\ & = (1-p) \text{tr} A_1 \prod_{\mu=1}^{N-1} \left(E_1^{n_\mu} D_1 + \frac{p}{1-p} \theta(n_\mu) E_1^{n_\mu-1} D_1 E_1 \right) E_1^{n_N} \\ & + \frac{\beta}{1-\beta} \text{tr} A_1 ((1-p)\delta_{n_1,0} + p\delta_{n_1,1} E_1) \prod_{\mu=2}^{N-1} \left(E_1^{n_\mu} D_1 + \frac{p}{1-p} \theta(n_\mu) E_1^{n_\mu-1} D_1 E_1 \right) E_1^{n_N} D_1 \\ & + p \text{tr} A_1 E_1 \prod_{\mu=1}^{N-1} \left(E_1^{n_\mu} D_1 + \frac{p}{1-p} \theta(n_\mu) E_1^{n_\mu-1} D_1 E_1 \right) E_1^{n_N-1} \theta(n_N). \end{aligned} \quad (5.53)$$

Before we now come to the proof rewrite (5.53) after multiplying both sides by $\prod_{\mu=2}^{N-1} (1 -$

$p\theta(n_\mu) \cdot (1 - \beta\delta_{n_N,0})/(1 - p\delta_{n_N,0})$. This leads to

$$\begin{aligned}
& \text{tr} A_1 \prod_{\mu=1}^{N-1} E_1^{n_\mu} D_1 E_1^{n_N} \\
&= [1 - \beta\delta_{n_N,0} - p\theta(n_N)] \text{tr} A_1 \prod_{\mu=1}^{N-1} \left[(1 - p\theta(n_\mu)) E_1^{n_\mu} D_1 + p\theta(n_\mu) E_1^{n_\mu-1} D_1 E_1 \right] E_1^{n_N} \\
&+ \beta \frac{1 - p\theta(n_N)}{1 - \beta\theta(n_N)} \text{tr} A_1 (\delta_{n_1,0} + p\delta_{n_1,1} E_1) \prod_{\mu=2}^{N-1} \left[(1 - p\theta(n_\mu)) E_1^{n_\mu} D_1 + p\theta(n_\mu) E_1^{n_\mu-1} D_1 E_1 \right] E_1^{n_N} D_1 \\
&+ p\theta(n_N) \text{tr} A_1 E_1 \prod_{\mu=1}^{N-1} \left[(1 - p\theta(n_\mu)) E_1^{n_\mu} D_1 + p\theta(n_\mu) E_1^{n_\mu-1} D_1 E_1 \right] E_1^{n_N-1}. \tag{5.54}
\end{aligned}$$

In Appendix A a formulation of the master equation in terms of the un-indexed operators is presented.

5.2.2 Proof of the matrix-product ansatz

We will prove this equation by case differentiation, considering different values of n_1 and n_N . We assume always $N \geq 2$, since the case $N = 1$ is trivial. For the proof the following simplification of the bulk terms under the product is essential:

$$\begin{aligned}
& (1 - p\theta(n_\mu)) E_1^{n_\mu} D_1 + p\theta(n_\mu) E_1^{n_\mu-1} D_1 E_1 \\
&= \delta_{n_\mu,0} D_1 + \theta(n_\mu) E_1^{n_\mu-1} [(1 - p) E_1 D_1 + p D_1 E_1] \\
&= \delta_{n_\mu,0} D_1 + (1 - p) \theta(n_\mu) E_1^{n_\mu-1} [E_1 D_1 + p(D_1 + E_1 + p)] \\
&= \delta_{n_\mu,0} D_1 + (1 - p) \theta(n_\mu) E_1^{n_\mu-1} [(E_1 + p) D_1 + p(E_1 + p)] \\
&= \delta_{n_\mu,0} D_1 + (1 - p) \theta(n_\mu) E_1^{n_\mu-1} (E_1 + p)(D_1 + p). \tag{5.55}
\end{aligned}$$

Here we have factors $(E_1 + p)(D_1 + p)$. Note that

$$(1 - p)(D_1 + p)(E_1 + p) = D_1 E_1. \tag{5.56}$$

This can be used to simplify the following equation which turns out to be the key to the proof:

$$\begin{aligned}
& (D_1 + p) \left[\delta_{n_\mu,0} D_1 + (1 - p) \theta(n_\mu) E_1^{n_\mu-1} (E_1 + p)(D_1 + p) \right] \\
&= \left[\delta_{n_\mu,0} D_1 + (1 - p) \theta(n_\mu) (D_1 + p) E_1^{n_\mu-1} (E_1 + p) \right] (D_1 + p) \\
&= \left[\delta_{n_\mu,0} D_1 + (1 - p) \theta(n_\mu) (D_1 + p)(E_1 + p) E_1^{n_\mu-1} \right] (D_1 + p) \\
&= \left[\delta_{n_\mu,0} D_1 + \theta(n_\mu) D_1 E_1^{n_\mu} \right] (D_1 + p) \\
&= D_1 E_1^{n_\mu} (D_1 + p). \tag{5.57}
\end{aligned}$$

Here we have used the fairly simple but essential commutation relations

$$E_1(E_1 + p) = (E_1 + p)E_1, \quad D_1(D_1 + p) = (D_1 + p)D_1. \tag{5.58}$$

As a consequence one has

$$(D_1 + p) \prod_{\mu} \left[(1 - p\theta(n_\mu)) E_1^{n_\mu} D_1 + p\theta(n_\mu) E_1^{n_\mu-1} D_1 E_1 \right] = \prod_{\mu} [D_1 E_1^{n_\mu}] (D_1 + p). \tag{5.59}$$

- $n_1 = 0$ and $n_N > 0$

In this case the master equation (5.54) becomes

$$\begin{aligned} \text{tr} A_1 \prod_{\mu=2}^N D_1 E_1^{n_\mu} &= (1-p) \text{tr} A_1 D_1 \prod_{\mu=2}^{N-1} \left[(1-p\theta(n_\mu)) E_1^{n_\mu} D_1 + p\theta(n_\mu) E_1^{n_\mu-1} D_1 E_1 \right] E_1^{n_N} \\ &+ \beta \frac{1-p}{1-\beta} \text{tr} D_1 A_1 \prod_{\mu=2}^{N-1} \left[(1-p\theta(n_\mu)) E_1^{n_\mu} D_1 + p\theta(n_\mu) E_1^{n_\mu-1} D_1 E_1 \right] E_1^{n_N} \\ &+ p \text{tr} A_1 E_1 D_1 \prod_{\mu=2}^{N-1} \left[(1-p\theta(n_\mu)) E_1^{n_\mu} D_1 + p\theta(n_\mu) E_1^{n_\mu-1} D_1 E_1 \right] E_1^{n_N-1}. \end{aligned} \quad (5.60)$$

Now use in the second term on the right-hand side (5.34) and in the third term (5.36) then this can be combined to

$$\begin{aligned} &\text{tr} A_1 \prod_{\mu=2}^N D_1 E_1^{n_\mu} \\ &= (1-p) \text{tr} A_1 (D_1 + p) \prod_{\mu=2}^{N-1} \left[(1-p\theta(n_\mu)) E_1^{n_\mu} D_1 + p\theta(n_\mu) E_1^{n_\mu-1} D_1 E_1 \right] E_1^{n_N-1} (E_1 + p). \end{aligned} \quad (5.61)$$

Now using (5.59) leads to

$$\text{tr} A_1 \prod_{\mu=2}^N D_1 E_1^{n_\mu} = (1-p) \text{tr} A_1 \prod_{\mu=2}^{N-1} [D_1 E_1^{n_\mu}] (D_1 + p) E_1^{n_N-1} (E_1 + p) \quad (5.62)$$

which reduces to an identity on using (5.56) and (5.58).

- $n_1 = n_N = 0$

The master equation becomes

$$p \text{tr} \prod_{\mu=2}^{N-1} D_1 E_1^{n_\mu} A_1 = \beta \text{tr} (D_1 + p) \prod_{\mu=2}^{N-1} \left[(1-p\theta(n_\mu)) E_1^{n_\mu} D_1 + p\theta(n_\mu) E_1^{n_\mu-1} D_1 E_1 \right] A_1. \quad (5.63)$$

Again using (5.59) leads to

$$p \text{tr} \prod_{\mu=2}^{N-1} D_1 E_1^{n_\mu} A_1 = \beta \text{tr} \prod_{\mu=2}^{N-1} [D_1 E_1^{n_\mu}] (D_1 + p) A_1. \quad (5.64)$$

Now, as a consequence of (5.34), one has $\beta(D_1 + p)A_1 = pA_1$, hence also this case is proven.

- $n_1 = 1, n_N = 0$

$$\begin{aligned} &\text{tr} A_1 E_1 \prod_{\mu=2}^{N-1} [D_1 E_1^{n_\mu}] D_1 \\ &= (1-\beta) A_1 [(1-p) E_1 D_1 + p D_1 E_1] \prod_{\mu=2}^{N-1} \left[(1-p\theta(n_\mu)) E_1^{n_\mu} D_1 + p\theta(n_\mu) E_1^{n_\mu-1} D_1 E_1 \right] \\ &+ p^2 (1-\beta) A_1 E_1 \prod_{\mu=2}^{N-1} \left[(1-p\theta(n_\mu)) E_1^{n_\mu} D_1 + p\theta(n_\mu) E_1^{n_\mu-1} D_1 E_1 \right]. \end{aligned} \quad (5.65)$$

Applying to the left-hand side (5.34) and rewriting $(1-p)E_1D_1 + pD_1E_1 = (1-p)(E_1+p)(D_1+p)$, transporting (D_1+p) through the chain leads to

$$\begin{aligned} p \operatorname{tr} A_1 E_1 \prod_{\mu=2}^{N-1} D_1 E_1^{n_\mu} &= \beta(1-p) \operatorname{tr}(D_1+p) A_1 (E_1+p) \prod_{\mu=2}^{N-1} D_1 E_1^{n_\mu} \\ &+ p^2 \beta \operatorname{tr} A_1 E_1 \prod_{\mu=2}^{N-1} \left[(1-p\theta(n_\mu)) E_1^{n_\mu} D_1 + p\theta(n_\mu) E_1^{n_\mu-1} D_1 E_1 \right]. \end{aligned} \quad (5.66)$$

Consider $A_1 E_1 \left[\delta_{n_2,0} D_1 + (1-p)\theta(n_2) E_1^{n_2-1} (E_1+p)(D_1+p) \right]$. This is for $n_2 = 0$, due to (5.36), equal to $(1-p)A_1(D_1+p)$. For $n_2 > 0$, due to (5.35), $(1-p)A_1 E_1^{n_2} (D_1+p)$. So, concluding one finds after combining both expressions

$$A_1 E_1 \left[\delta_{n_2,0} D_1 + (1-p)\theta(n_2) E_1^{n_2-1} (E_1+p)(D_1+p) \right] = (1-p) A_1 E_1^{n_2} (D_1+p). \quad (5.67)$$

Now the (D_1+p) -term can be transported through the chain according to (5.59). Finally use $\beta(D_1+p)A_1 = pA_1$. Then

$$\begin{aligned} \operatorname{tr} A_1 E_1 \prod_{\mu=2}^{N-1} D_1 E_1^{n_\mu} &= (1-p) \operatorname{tr} A_1 (E_1+p) D_1 E_1^{n_2} \prod_{\mu=3}^{N-1} D_1 E_1^{n_\mu} \\ &+ p^2 (1-p) \operatorname{tr} A_1 E_1^{n_2} \prod_{\mu=3}^{N-1} D_1 E_1^{n_\mu}. \end{aligned} \quad (5.68)$$

Now using (5.36) leads to the required identity.

- $n_1 = 1, n_N > 0$

We start with

$$\begin{aligned} &\operatorname{tr} A_1 E_1 \prod_{\mu=2}^N D_1 E_1^{n_\mu} \\ &= (1-p) A_1 [(1-p)E_1 D_1 + pD_1 E_1] \prod_{\mu=2}^{N-1} \left[(1-p\theta(n_\mu)) E_1^{n_\mu} D_1 + p\theta(n_\mu) E_1^{n_\mu-1} D_1 E_1 \right] E_1^{n_N} \\ &+ p^2 (1-p) A_1 E_1 \prod_{\mu=2}^{N-1} \left[(1-p\theta(n_\mu)) E_1^{n_\mu} D_1 + p\theta(n_\mu) E_1^{n_\mu-1} D_1 E_1 \right] E_1^{n_N} \\ &+ p A_1 E_1 [(1-p)E_1 D_1 + pD_1 E_1] \prod_{\mu=2}^{N-1} \left[(1-p\theta(n_\mu)) E_1^{n_\mu} D_1 + p\theta(n_\mu) E_1^{n_\mu-1} D_1 E_1 \right] E_1^{n_N-1}, \end{aligned} \quad (5.69)$$

which is obtained from the master equation by once applying (5.34) to the second term on the r.h.s. Now follow the same steps as in the proof for $n_1 = 1$ and $n_N = 0$.

One arrives at

$$\begin{aligned}
\text{tr} A_1 E_1 \prod_{\mu=2}^N D_1 E_1^{n_\mu} &= (1-p)^2 \text{tr} A_1 (E_1 + p) D_1 E_1^{n_2} \prod_{\mu=3}^{N-1} [D_1 E_1^{n_\mu}] (D_1 + p) E_1^{n_N} \\
&+ p^2 (1-p)^2 \text{tr} A_1 E_1^{n_2} \prod_{\mu=3}^{N-1} [D_1 E_1^{n_\mu}] (D_1 + p) E_1^{n_N} \quad (5.70) \\
&+ p(1-p) \text{tr} A_1 E_1 D_1 E_1^{n_2} \prod_{\mu=3}^{N-1} [D_1 E_1^{n_\mu}] (D_1 + p) E_1^{n_N-1}.
\end{aligned}$$

Combining the first and second term on the r.h.s. with the help of (5.36) gives

$$\begin{aligned}
\text{tr} A_1 E_1 \prod_{\mu=2}^N D_1 E_1^{n_\mu} &= (1-p) \text{tr} A_1 E_1 \prod_{\mu=2}^{N-1} [D_1 E_1^{n_\mu}] (D_1 + p) E_1^{n_N} \\
&+ p(1-p) \text{tr} A_1 E_1 \prod_{\mu=2}^{N-1} [D_1 E_1^{n_\mu}] (D_1 + p) E_1^{n_N-1} \quad (5.71) \\
&= (1-p) \text{tr} A_1 E_1 \prod_{\mu=2}^{N-1} [D_1 E_1^{n_\mu}] (D_1 + p) E_1^{n_N-1} (E_1 + p).
\end{aligned}$$

Due to (5.56) both sides are the same.

- $n_1 \geq 2, n_N = 0$

The master equation reads after using (5.59) and writing the matrix product under the trace conveniently

$$\text{tr} D_1 A_1 E_1^{n_1} \prod_{\mu=2}^{N-1} D_1 E_1^{n_\mu} = (1-p)(1-\beta) \text{tr} (D_1 + p) A_1 E_1^{n_1-1} (E_1 + p) \prod_{\mu=2}^{N-1} [D_1 E_1^{n_\mu}]. \quad (5.72)$$

Consider $(1-p) A_1 E_1^{n_1-1} (E_1 + p)$. Since $n_1 \geq 2$ equation (5.35) can be applied and leads to $(1-p) A_1 E_1^{n_1-1} (E_1 + p) = A_1 E_1^{n_1}$. Then it remains to show that $(1-\beta)(D_1 + p) = D_1 A_1$ and this is a consequence of (5.34).

- $n_1 \geq 2, n_N > 0$

The master equation reads

$$\begin{aligned}
& \text{tr} A_1 E_1^{n_1} \prod_{\mu=2}^N D_1 E_1^{n_\mu} \\
&= (1-p) \text{tr} A_1 \left[(1-p) E_1^{n_1} D_1 + p E_1^{n_1-1} D_1 E_1 \right] \\
& \quad \prod_{\mu=2}^{N-1} \left[(1-p \theta(n_\mu)) E_1^{n_\mu} D_1 + p \theta(n_\mu) E_1^{n_\mu-1} D_1 E_1 \right] E_1^{n_N} \\
&+ p \text{tr} A_1 E_1 \left[(1-p) E_1^{n_1} D_1 + p E_1^{n_1-1} D_1 E_1 \right] \\
& \quad \prod_{\mu=2}^{N-1} \left[(1-p \theta(n_\mu)) E_1^{n_\mu} D_1 + p \theta(n_\mu) E_1^{n_\mu-1} D_1 E_1 \right] E_1^{n_N-1} \\
&= (1-p)^2 \text{tr} A_1 E_1^{n_1-1} (E_1 + p) \prod_{\mu=2}^{N-1} [D_1 E_1^{n_\mu}] (D_1 + p) E_1^{n_N} \\
&+ p(1-p) \text{tr} A_1 E_1^{n_1} (E_1 + p) \prod_{\mu=2}^{N-1} [D_1 E_1^{n_\mu}] (D_1 + p) E_1^{n_N-1}. \tag{5.73}
\end{aligned}$$

Since $n_1 \geq 2$ again equation (5.35) can be applied to give

$$\text{tr} A_1 E_1^{n_1} \prod_{\mu=2}^N D_1 E_1^{n_\mu} = (1-p) \text{tr} A_1 E_1^{n_1} \prod_{\mu=2}^{N-1} [D_1 E_1^{n_\mu}] (D_1 + p) (E_1 + p) E_1^{n_N-1}. \tag{5.74}$$

Now once using (5.56) completes the proof.

5.3 Calculation of the normalization-generating function

In contrast to the open boundary ASEP we have to deal with a fixed number of particles and holes. So we have to perform the calculation grand-canonically. We introduce a fugacity x for particles and a fugacity y for hole pairs. The single hole is weighted accordingly by \sqrt{y} . The grand-canonical normalization can be written as

$$\mathcal{Z}_{N+M} = \langle W | C(x, y)^{N+M} | V \rangle, \tag{5.75}$$

and the canonical function as

$$Z_{N,2M+1} = \{x^N\} \{y^{M+1/2}\} x \sqrt{y} \langle W | C(x, y)^{N+M} | V \rangle. \tag{5.76}$$

To calculate (5.75) we need an expression for

$$C^n(x, y) = (xD + yE)^n = \begin{pmatrix} yE_1 & x(1-p)^{-1}D_1 \\ yE_1 & xD_1 \end{pmatrix}^n. \tag{5.77}$$

To use results from the calculation of Evans, Rajewsky and Speer [24] for the parallel ASEP with open boundaries we would prefer C in the form

$$\tilde{C}(x, y) = \begin{pmatrix} xD_1 + yE_1 & \sqrt{\frac{p}{1-p}} xD_1 \\ \sqrt{\frac{p}{1-p}} yE_1 & 0 \end{pmatrix}. \tag{5.78}$$

In fact there is a similarity transform $C = R^{-1}\tilde{C}R$ that puts C into the form of \tilde{C} with transformation matrices

$$R = \begin{pmatrix} (p(1-p))^{1/2} & 0 \\ -(1-p) & 1 \end{pmatrix}, R^{-1} = \begin{pmatrix} (p(1-p))^{-1/2} & 0 \\ ((1-p)/p)^{1/2} & 1 \end{pmatrix}. \quad (5.79)$$

Then (5.75) turns into

$$\mathcal{Z}_{N+M} = \langle W | R^{-1} \tilde{C}(x, y)^{N+M} R | V \rangle. \quad (5.80)$$

Considering \tilde{C}^n for small values of n one is eventually drawn to the following formula

$$\tilde{C}^n(x, y) = \begin{pmatrix} G(n) & \sqrt{\frac{p}{1-p}} x G(n-1) D_1 \\ \sqrt{\frac{p}{1-p}} y E_1 G(n-1) & xy \frac{p}{1-p} E_1 G(n-2) D_1 \end{pmatrix}, \quad n \geq 1. \quad (5.81)$$

Before we define the functions $G(n)$ consider the action of R and R^{-1} on the boundary vectors:

$$\langle W | R^{-1} = \left(\sqrt{\frac{1-p}{p}} \langle W_1 |, \langle W_1 | \right), \quad R | V = \left(\sqrt{p(1-p)} | V_1 \rangle, \frac{\beta(1-p)}{1-\beta} | V_1 \rangle \right)^t. \quad (5.82)$$

Using this together with (5.81), we find for $n \geq 1$

$$\begin{aligned} \langle W | R^{-1} \tilde{C}^n R | V \rangle &= (1-p) \langle W_1 | G(n) | V_1 \rangle + py \langle W_1 | E_1 G(n-1) | V_1 \rangle \\ &+ px [(1-p) \langle W_1 | G(n-1) | V_1 \rangle + py \langle W_1 | E_1 G(n-2) | V_1 \rangle]. \end{aligned} \quad (5.83)$$

On the r.h.s. we define

$$S_n = (1-p) \langle W_1 | G_n(x, y) | V_1 \rangle + py \langle W_1 | E_1 G_{n-1}(x, y) | V_1 \rangle, \quad (5.84)$$

where $S_0 = (1-p)(1-\beta)$, so that the grand-canonical normalization becomes

$$\mathcal{Z}_n = S_n + px S_{n-1}, \quad n \geq 1. \quad (5.85)$$

Now to the functions $G(n)$. They obey the following recursions

$$G(n) = C_1 G(n-1) + pxy K G(n-2), \quad (5.86)$$

$$G(n) = G(n-1) C_1 + pxy G(n-2) K, \quad (5.87)$$

$$\text{with } G(-1) := 0 \text{ and } G(0) = 1, \quad (5.88)$$

so that $G(1) = C_1$, $G(2) = C_1^2 + pxy K$ and so on. For $n = 1$ equation (5.81) directly turns into (5.78) if one inserts the values for $G(1)$, $G(0)$ and $G(-1)$ from (5.88). Now we prove (5.81) by induction. Assume that it is true up to a value n . Then

$$\begin{aligned} \tilde{C}^n \tilde{C} &= \begin{pmatrix} G(n) & \sqrt{\frac{p}{1-p}} x G(n-1) D_1 \\ \sqrt{\frac{p}{1-p}} y E_1 G(n-1) & xy \frac{p}{1-p} E_1 G(n-2) D_1 \end{pmatrix} \begin{pmatrix} C_1 & \sqrt{\frac{p}{1-p}} x D_1 \\ \sqrt{\frac{p}{1-p}} y E_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} G(n) C_1 + pxy G(n-1) K & \sqrt{\frac{p}{1-p}} x G(n) D_1 \\ \sqrt{\frac{p}{1-p}} y E_1 [G(n-1) C_1 + pxy G(n-2) K] & xy \frac{p}{1-p} E_1 G(n-1) D_1 \end{pmatrix} \end{aligned} \quad (5.89)$$

where we have used $(1-p)^{-1}D_1E_1 = K$. Now using the second recursion (5.87) for $G(n)$ completes the proof. Of course the first recursion (5.86) can be proven in the same way by considering $\tilde{C}\tilde{C}^n$. Iterating (5.86) yields

$$G_{xy}(n) = \sum_{l=0}^{\infty} (pxy)^l \left[\prod_{i=0}^{n-l} \sum_{\sigma_i=0,1} (C_1\delta_{\sigma_i,0} + K\delta_{\sigma_i,1}) \cdot \delta_{\sum \sigma_i, l} \right], \quad (5.90)$$

where the term in the parenthesis denotes the sum over all matrix products of $C_1 = xD_1 + yE_1$ and $K = D_1 + E_1 + p$ with exactly l K s and $n - 2l$ C_1 s. Of course one has $l \leq n - l$.

This gives

$$\begin{aligned} G_0(x, y) &= 1, \\ G_1(x, y) &= C_1, \\ G_2(x, y) &= C_1^2 + pxyK, \\ G_3(x, y) &= C_1^3 + pxy(C_1K + KC_1), \\ G_4(x, y) &= C_1^4 + pxy(C_1^2K + C_1KC_1 + KC_1^2) + (pxy)^2K^2, \\ G_5(x, y) &= C_1^5 + pxy(C_1^3 + C_1^2KC_1 + C_1KC_1^2 + KC_1^3) \\ &\quad + (pxy)^2(C_1K^2 + KC_1K + K^2C_1) \\ &\quad \dots \end{aligned} \quad (5.91)$$

It turns out to be rather difficult to work directly with $G_n(x, y)$. Instead we will consider the generating function:

$$\mathcal{F}(x, y, \lambda) = \sum_{n=0}^{\infty} \lambda^n G_n(x, y). \quad (5.92)$$

If one multiplies (5.91) by λ^n and sorts the result depending on the lengths of words that occur, one obtains

$$\mathcal{F}(x, y, \lambda) = \sum_{n=0}^{\infty} \lambda^n (C_1 + pxy\lambda K)^n. \quad (5.93)$$

The term under the sum is $C_1 + pxy\lambda K = (x + pxy\lambda)D_1 + (y + pxy\lambda)E_1 + p^2xy\lambda$. The aim of our calculation is to get rid of the matrices at one point and it is very convenient to transform the matrices [15]. Define

$$D_1 = \sqrt{\frac{y + pxy\lambda}{x + pxy\lambda}} [D'_1 - (1-p)] + 1-p, \quad (5.94)$$

$$E_1 = \sqrt{\frac{x + pxy\lambda}{y + pxy\lambda}} [E'_1 - (1-p)] + 1-p. \quad (5.95)$$

One can check that these primed matrices indeed fulfill

$$D'_1E'_1 = (1-p)(D'_1 + E'_1 + p) = (1-p)K'. \quad (5.96)$$

In this notation $C_1 + pxy\lambda K$ becomes finally

$$C_1 + pxy\lambda K = \sqrt{(x + pxy\lambda)(y + pxy\lambda)}K' + \omega, \quad (5.97)$$

with

$$\omega = \omega(x, y, \lambda) = \sqrt{(x + pxy\lambda)(y + pxy\lambda)}(p - 2) + (1 - p)(x + y + 2pxy\lambda) + p^2xy\lambda. \quad (5.98)$$

Executing the sum in (5.93) and a little development of the result yields

$$\mathcal{F}(x, y, \lambda) = (1 - \omega\lambda)^{-1} \left(1 - \frac{\lambda\sqrt{(x + pxy\lambda)(y + pxy\lambda)}}{1 - \omega\lambda} K' \right)^{-1}. \quad (5.99)$$

Now we write

$$1 - \frac{\lambda\sqrt{(x + pxy\lambda)(y + pxy\lambda)}}{1 - \omega\lambda} K' = \left(1 - \frac{\gamma}{1 - p} D'_1 \right) \left(1 - \frac{\gamma}{1 - p} E'_1 \right) \frac{1 - p}{1 - p(1 - \gamma)} \quad (5.100)$$

in terms of a function γ to be determined. Working this out using (5.96) one is drawn to the following condition that defines γ :

$$\frac{\gamma(1 - \gamma)}{1 - p(1 - \gamma)} = \frac{\lambda\sqrt{(x + pxy\lambda)(y + pxy\lambda)}}{1 - \omega\lambda}. \quad (5.101)$$

A similar trick has also be used in [15]. We conclude that

$$\left[1 - \frac{\lambda\sqrt{(x + pxy\lambda)(y + pxy\lambda)}}{1 - \omega\lambda} K' \right]^{-1} = \left(1 - \frac{\gamma}{1 - p} E'_1 \right)^{-1} \left(1 - \frac{\gamma}{1 - p} D'_1 \right)^{-1} \left(1 + \frac{p}{1 - p} \gamma \right). \quad (5.102)$$

Finally we have

$$\mathcal{F}(x, y, \lambda) = \frac{1 + \frac{p}{1 - p} \gamma}{1 - \omega\lambda} \left(1 - \frac{\gamma}{1 - p} E'_1 \right)^{-1} \left(1 - \frac{\gamma}{1 - p} D'_1 \right)^{-1}. \quad (5.103)$$

From this the generating function of the S_n can be calculated:

$$\mathcal{S}(\lambda) = \sum_{n=0}^{\infty} \lambda^n S_n = \langle W_1 | (1 - p + py\lambda E_1) \mathcal{F} | V_1 \rangle. \quad (5.104)$$

To calculate this we need an expression for the action of D'_1 and E'_1 on the boundary vectors. For this auxiliary calculation we write shorthand $z := pxy\lambda$.

$$\begin{aligned} D_1'^q(x, y, z) | V_1 \rangle &= \left[\sqrt{\frac{x+z}{y+z}} D_1 + (1 - p) \left(1 - \sqrt{\frac{x+z}{y+z}} \right) \right]^q | V_1 \rangle \\ &= \sum_{k=0}^q \binom{q}{k} \sqrt{\frac{x+z}{y+z}}^k \left[(1 - p) \left(1 - \sqrt{\frac{x+z}{y+z}} \right) \right]^{q-k} D_1^k | V_1 \rangle \\ &= \sum_{k=0}^q \binom{q}{k} \left(\sqrt{\frac{x+z}{y+z}} \frac{p(1 - \beta)}{\beta} \right)^k \left[(1 - p) \left(1 - \sqrt{\frac{x+z}{y+z}} \right) \right]^{q-k} | V_1 \rangle \\ &= \dots \\ &= \left(\sqrt{\frac{x+z}{y+z}} \frac{p - \beta}{\beta} + 1 - p \right)^q | V_1 \rangle. \end{aligned} \quad (5.105)$$

Once having this we can directly calculate $\langle W_1 | E_1'^r | V_1 \rangle$ which is done in a similar fashion. Now one needs the relation $\langle W_1 | E_1^k | V_1 \rangle = (1 - p)^k / (1 - \beta) \langle W_1 | V_1 \rangle$ for $k \geq 1$. This leads to

$$\langle W_1 | E_1'^r | V_1 \rangle = (1 - p)^r \left[1 - \beta \left(1 - \sqrt{\frac{y+z}{x+z}} \right)^r \right]. \quad (5.106)$$

where we have chosen $\langle W_1|V_1\rangle = 1 - \beta$ which coincides with our representation. Similarly one finds

$$\langle W_1|E_1 E_1^r|V_1\rangle = (1 - p)^{r+1}. \quad (5.107)$$

Now from this the matrix element of the generating function of the S_n can be calculated:

$$\mathcal{S} = \frac{1 - p + p\gamma}{1 - \omega\lambda} \frac{1}{1 - \gamma \left(1 + \frac{p - \beta}{\beta(1 - p)} \sqrt{\frac{x + pxy\lambda}{y + pxy\lambda}}\right)} \left[\frac{1 + py\lambda}{1 - \gamma} - \frac{\beta}{1 - \gamma \left(1 - \sqrt{\frac{y + pxy\lambda}{x + pxy\lambda}}\right)} \right] \quad (5.108)$$

At this stage it is helpful to set $y := 1$, since the generating function of the process is fully determined by x and λ . Then the generating function of the grand-canonical partition function is due to (5.85) defined by

$$\mathcal{Z}(\lambda, x, y) = (1 + px\lambda)\mathcal{S}(\lambda, x, y). \quad (5.109)$$

This expression will be used to calculate the asymptotic form of the grand-canonical function. For small systems one can work directly with (5.85), which gives for $n = 1$:

$$\mathcal{Z}_1 = S_1 + pxS_0 = (1 - p)x\langle W_1|D_1|V_1\rangle + y\langle W_1|E_1|V_1\rangle + xp(1 - p)\langle W_1|V_1\rangle, \quad (5.110)$$

which leads with (5.76) and $\langle W_1|V_1\rangle = 1 - \beta$ to the canonical functions $Z_{2,1} = p(1 - p)(1 - \beta)/\beta$ and $Z_{1,3} = 1 - p$. For $n = 2$ the result can be written as

$$\begin{aligned} \mathcal{Z}_2 &= (1 - p)x^2\langle W_1|D_1^2|V_1\rangle + y^2\langle W_1|E_1^2|V_1\rangle + xy\langle W_1|D_1 E_1|V_1\rangle + xy\langle W_1|E_1 D_1|V_1\rangle \\ &+ p(1 - p)x^2\langle W_1|D_1|V_1\rangle + pxy\langle W_1|E_1|V_1\rangle. \end{aligned} \quad (5.111)$$

The coefficient of $x \cdot y$ gives the canonical function $Z_{2,3} = 1 - p[2p(1 - \beta) + \beta]/\beta$ and so on. As a test, the same results can be obtained by adding all the possible matrix-product states for given N and M and simplifying using the algebraic rules.

5.4 Limits

5.4.1 Random sequential

The result for random-sequential dynamics can be recovered from replacing β by $p\beta$ and then taking the limit $p \rightarrow 0$. Then (5.108) turns with $y = 1$ into

$$\langle W_1|\mathcal{S}(\lambda)|V_1\rangle = \frac{1}{1 - \omega\lambda} \frac{1}{1 - \gamma/\beta'} \frac{1}{1 - \gamma}, \quad (5.112)$$

with

$$\omega = \omega(x) = \frac{1}{(1 - \sqrt{x})^2}, \quad (5.113)$$

$$\gamma = \gamma(x, \lambda) \text{ to be determined from } \gamma(1 - \gamma) = \frac{\lambda\sqrt{x}}{1 - \lambda(1 - \sqrt{x})^2}, \quad (5.114)$$

$$1/\beta' = 1 + \frac{1 - \beta}{\beta}\sqrt{x}, \quad (5.115)$$

which coincides with the result in [15] for the generating function of the grand-canonical normalization of the defect ASEP with $\alpha = 1$.

5.4.2 ASEP with open boundaries and parallel update

In the ASEP with open boundaries and parallel update the number of particles and holes is not fixed so that one can set $x = y = 1$ for the associated fugacities. In this case one does not have to go the way over the primed matrices and can calculate directly an expression for the power K^n as it was done in [24]. However we can recover this result from our calculation. In fact the primed matrices defined through (5.94) turn into the normal matrices for $x = y = 1$ and ω defined in (5.98) simply turns into $\omega = -p$. Using this, equation (5.99) becomes

$$\mathcal{F} = (1 + p\lambda)^{-1}(1 - K)^{-1}. \quad (5.116)$$

Writing this as the product of two geometric series and working out the convolution one immediately finds

$$\mathcal{F} = \sum_{n=0}^{\infty} G_n \lambda^n = \sum_{n=0}^{\infty} \sum_{k=0}^n (-p)^k K^{n-k} \lambda^n, \quad (5.117)$$

so that

$$G_n = \sum_{k=0}^n (-p)^k K^{n-k}, \quad (5.118)$$

which is the result derived in [24]. So this limit is properly contained in the generating function (5.99). Then starting from (5.80) with the different boundary vectors for this model, namely $\langle W|R^{-1}$ replaced by $(\langle W_1|, \langle W_1|\sqrt{p/(1-p)})$ and $|V-1\rangle$ replaced by $(|V_1\rangle, \beta/(1-\beta)\sqrt{(1-p)/p})^t$ leads to the normalization

$$\mathcal{Z}_L = S_n + pS_{n-1}, \quad (5.119)$$

with

$$S_n = \langle W_1|G_n|V_1\rangle + p\langle W_1|G_{n-1}|V_1\rangle. \quad (5.120)$$

Note that S_n is the same as z_n in [24].

5.5 Asymptotics from the generating function

- First phase

We start with the pole at

$$\gamma(x, \lambda_1) = \left[1 + \frac{p - \beta}{\beta(1 - p)} \sqrt{\frac{x + px\lambda}{1 + px\lambda}} \right]^{-1}. \quad (5.121)$$

It turns out that the resulting equation in p , x and λ_1 is easier to solve for x . The physically relevant solution for x in terms of λ_1 is

$$x = \frac{\beta}{p(1 - \beta)\lambda} \frac{\beta - p + p(1 - p)\lambda}{\beta - p - p^2\lambda}. \quad (5.122)$$

- Second phase

There is also a square-root singularity in the expression for γ :

$$\gamma = \frac{1}{2} \left[1 - pb \pm \sqrt{(1 + pb)^2 - 4b} \right], \text{ with } b = \frac{\lambda \sqrt{x(1 + p\lambda)(1 + px\lambda)}}{1 - \omega\lambda} \quad (5.123)$$

at

$$b_2^\pm = \frac{2 - p \pm 2\sqrt{1 - p}}{p^2}. \quad (5.124)$$

Surprisingly this condition can be written after some amount of algebra explicitly for x , which gives, choosing the physically relevant sign:

$$x = \left(\frac{1 - \sqrt{(1 - p)/\lambda(1 + p\lambda)}}{1 - p(1 + p\lambda)} \right)^2. \quad (5.125)$$

- Relation to the density

At first we introduce a formal density $\tilde{\rho}$ in the normal ASEP picture, where each matrix E represents a single hole. Then we have the standard relation

$$\tilde{\rho}_0 = -x \frac{\partial \ln \lambda(x)}{\partial x}, \quad (5.126)$$

with λ being the dominating singularity (either λ_1 or λ_2) and ρ the associated density. Keeping this in mind we not use a different symbol for λ being the dominant singularity. Instead of having a relation $\lambda(x)$, we have $x(\lambda)$, so rewriting this result yields

$$\lambda(x) = \frac{-x(\lambda)}{\rho x'(\lambda)}. \quad (5.127)$$

5.5.1 The phase transition

Equating relations (5.122) and (5.125) leads to an expression for λ at the phase transition:

$$\lambda_c = \frac{(p - \beta)^2}{p^2(1 - p)}. \quad (5.128)$$

Relating this to the density gives

$$\rho_c = \frac{\beta(1 - \beta)}{p - \beta^2}. \quad (5.129)$$

So this is the expression for the critical density where the phase transition takes place. This leads to the phase diagram, depicted in figure 5.1. The diagram shows additionally the line on which the velocity of the excess hole changes its sign which is calculated later. At first one needs expressions for the occupations around the excess.

For calculations of asymptotic quantities it is important to note that the asymptotic form of S_n is always

$$S_n \sim \lambda^{-n} \quad (5.130)$$

which follows from the theory of generating functions [15, 78]. This implies through (5.109) that the grand-canonical partition function scales as

$$\mathcal{Z}_n \sim (1 + px\lambda)\lambda^{-n}. \quad (5.131)$$

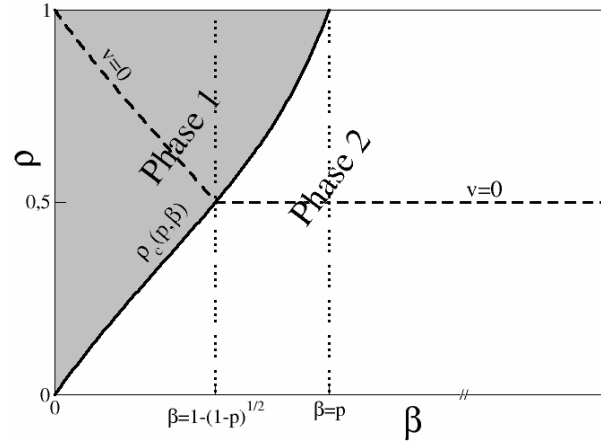


Figure 5.1: Phase diagram in terms of the density ρ and β for a fixed value of p , $p \in (0, 1)$. The thick line ρ_c given by (5.129) shows the phase-transition. On the dashed line, the velocity of the excess hole becomes zero. The special form of the curves though varies with p . The dotted curves serve as a guide to the eye.

5.5.2 Occupation behind the defect

At first we are going to calculate the probability of finding a particle directly behind the defect, which we denote by ρ_- :

$$\rho_- = \lim_{N+M \rightarrow \infty} \frac{x \langle W | C^{N+M-1} D | V \rangle}{\langle W | C^{N+M} | V \rangle}. \quad (5.132)$$

We start with the nominator

$$\begin{aligned} \langle W | C^n D | V \rangle &= \langle W | C^{n-1} x D D | V \rangle + \langle W | C^{n-1} y E D | V \rangle \\ &= \frac{p(1-\beta)}{\beta} \langle W | C^{n-1} x D | V \rangle + \frac{p}{\beta} \langle W | C^{n-1} y E | V \rangle \\ &= \frac{p}{\beta} \langle W | C^n | V \rangle - p x \langle W | C^{n-1} D | V \rangle. \end{aligned} \quad (5.133)$$

Comparing both sides of this equation with (5.85) one immediately finds

$$\langle W | C^{n-1} D | V \rangle = \frac{p}{\beta} S_{n-1}. \quad (5.134)$$

So that finally from (5.130) and (5.131)

$$\rho_- = \frac{p x \lambda}{\beta(1 + p x \lambda)}. \quad (5.135)$$

This gives in phase 1:

$$\rho_{-,1} = \frac{(1-p)(1 - \sqrt{1 - 4p\rho(1-\rho)})}{(p-\beta)\sqrt{1 - 4p\rho(1-\rho)} - 2p[\beta(1-\rho) - \rho] + p + \beta}. \quad (5.136)$$

In phase 2 the result can be written as

$$\rho_{-,2} = \frac{p}{\beta(1 + p\lambda_2)} \left(\frac{\sqrt{\lambda} - \sqrt{1-p}(1 + p\lambda_2)}{\sqrt{1-p} - p\sqrt{\lambda_2}} \right)^2. \quad (5.137)$$

As an example take p and β such that $\rho_c = 1/2$, namely $p = 3/4$ and $\beta = 1/2$. In figure (5.2) one sees how the curves corresponding to the two phases fit together to the dotted curve coming from a computer simulation with $L = 1000$. For $\rho < 1/2$ the system is in phase 2 and for $\rho > 1/2$ it is in phase 1. One sees that in the exact solution for $L \rightarrow \infty$ there is a discontinuity at $\rho = 1/2$.

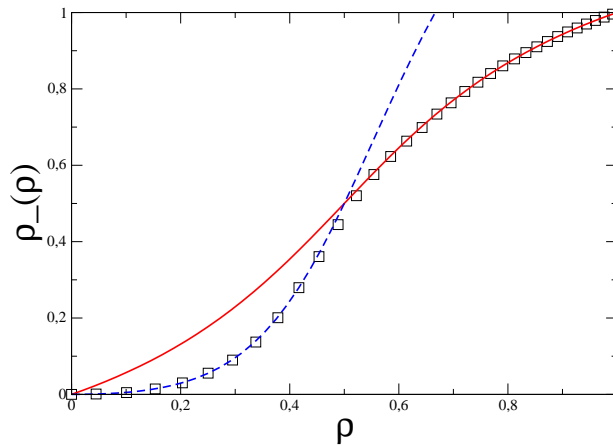


Figure 5.2: The solid curves show the two solutions for $\rho_-(\rho)$ in the two phases in comparison with the squares coming from a computer simulation. The hopping probabilities are $p = 3/4$ and $\beta = 1/2$, so that the critical density is $\rho_c = 1/2$. The system size is $L = 1000$.

For $\beta > p$ the probability ρ_- jumps at density $\rho = 1$ from p/β to 1, see figure 5.4. This can be explained as follows. If there is only one hole in an infinite system then ρ_- is trivially 1. But if there are three holes, both the hole pair and the single hole move backwards in space, however with $\beta > p$, the single hole moving faster, it approaches the hole pair against the direction of their motion. Then they form a cluster of three with probability p/β . As another example where one can see nicely how the two phases match together we take $p = 9/10$ and $\beta = 3/10$. Here the critical density is at $7/27 \approx 0.26$. The range of the ρ_- axis is truncated at $\rho_- = 1$ to limit the curve representing the solution of phase 2 in its unphysical region to the right of the transition, see figure (5.3).

For $p = \beta$ the system is completely in phase 2. The comparison between computer simulation and exact solution is shown in figure 5.4. This case corresponds to the right vertical dotted line in figure 5.1.

5.5.3 Occupation in front of the defect

Now come to the probability for the occupation in front of the 01-pair:

$$\rho_+(n) = \frac{x \langle W | DC^{n-1} | V \rangle}{\langle W | C^n | V \rangle} = 1 - \frac{y \langle W | EC^{n-1} | V \rangle}{\langle W | C^n | V \rangle}. \quad (5.138)$$

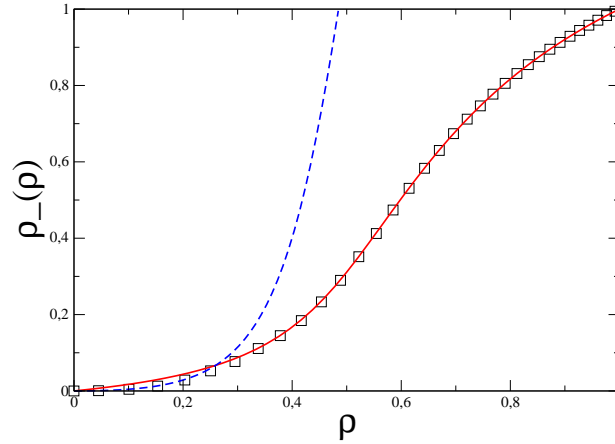


Figure 5.3: $\rho_-(\rho)$ in the two phases in comparison with computer simulation for $p = 9/10$ and $\beta = 3/10$. The system size is $L = 1000$.

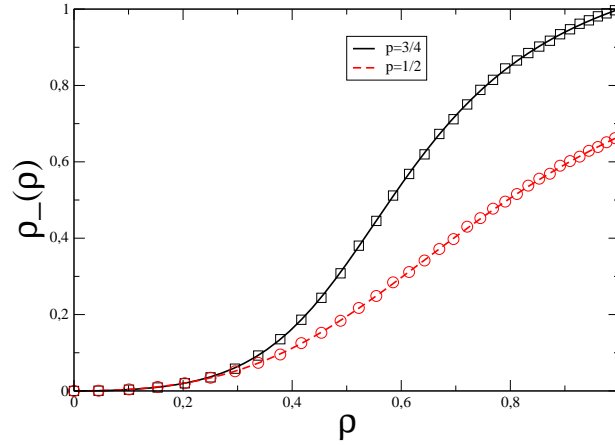


Figure 5.4: $\rho_-(\rho)$ for $\beta = 3/4$ and system size $L = 1000$. Dashed and continuous line: solution of phase 2 for $p = 1/2$ and $p = 3/4$ respectively. The symbols show the computer simulation.

Start with the nominator

$$\begin{aligned}
 \langle W|EC^n|V \rangle &= \langle W|ExDC^{n-1}|V \rangle + \langle W|EyEC^{n-1}|V \rangle \\
 &= x\langle W|(D+p)C^{n-1}|V \rangle + y(1-p)\langle W|EC^{n-1}|V \rangle \\
 &= \langle W|C^n|V \rangle + px\langle W|C^{n-1}|V \rangle - py\langle W|EC^{n-1}|V \rangle.
 \end{aligned} \tag{5.139}$$

Now define $T_n := \langle W|EC^n|V \rangle$. Then one has

$$T_n + pyT_{n-1} = Z_n + pxZ_{n-1} \tag{5.140}$$

and we conclude that for n large T_n scales as $\lambda^{-n} \cdot (1 + px\lambda)^2 / (1 + py\lambda)$. Thus

$$\rho_+ = 1 - y\lambda \frac{1 + px\lambda}{1 + py\lambda}. \tag{5.141}$$

This leads in phase 1 to

$$1 - \rho_{+,1} = \frac{p - \beta}{p^2(1 - \beta)} \frac{2p(1 - \rho) - 1 + \sqrt{1 - 4p\rho(1 - \rho)}}{1 - 2\rho + \sqrt{1 - 4p\rho(1 - \rho)}} \tag{5.142}$$

and in phase 2 in terms of λ simply:

$$1 - \rho_{+,2} = \lambda^2 \left(\frac{p - \sqrt{(1-p)/\lambda}}{1 - p(1 + p\lambda)} \right)^2. \quad (5.143)$$

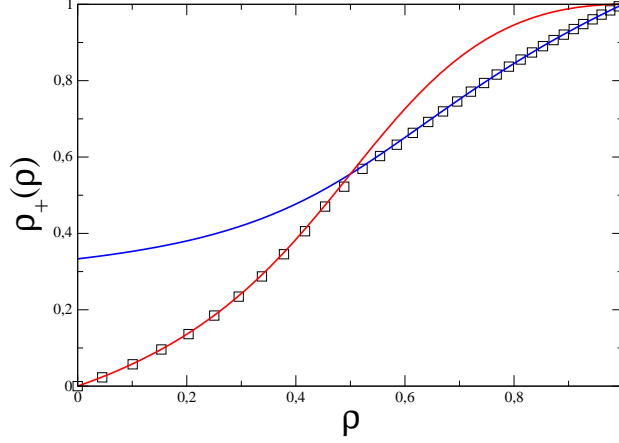


Figure 5.5: The solid curves show the two solutions for $\rho_+(\rho)$ in the two phases in comparison with the dotted curve coming from a computer simulation for $p = 3/4$ and $\beta = 1/2$, so that the critical density is $\rho_c = 1/2$. The system size is $L = 1000$.

Comparing the relations for ρ_- and $1 - \rho_+$ in both phases one sees that due to the broken particle-hole symmetry there is no proper symmetry between the two relations.

5.5.4 The defect velocity

The velocity of the defect in the two phases can be obtained by

$$v = p(1 - \rho_+)(1 - \beta\rho_-) - \beta\rho_-. \quad (5.144)$$

To be precise, this is the velocity of the single excess hole. If it has a particle directly behind it jumps backwards with probability β which leads to the second contribution $-\beta\rho_-$. If it has no particle in front (probability $(1 - \rho_+)$) it can jump forward with probability p unless it also has a particle behind which moves forward with probability β . This leads to the first contribution $p(1 - \rho_+)(1 - \beta\rho_-)$. Note that we always argue in terms of a density ρ that treats the hole pairs as single holes. Using (5.135) and (5.141) gives rise to the following expression for v :

$$v = \frac{py\lambda}{1 + py\lambda} - \frac{px\lambda}{1 + px\lambda} = \frac{p\lambda(y - x)}{(1 + px\lambda)(1 + py\lambda)}. \quad (5.145)$$

One sees that, due to the symmetry in x and y , the average defect velocity vanishes for equal densities of particles and hole pairs ($v(\rho = 1/2) = 0$). In phase 1 (5.145) is rewritten as

$$v_1(\rho) = \frac{p(p - \beta)(1 - \rho) - p(1 - \beta)J}{p(1 - \beta)(1 - \rho) - (p - \beta)J}, \quad (5.146)$$

where J is the total particle current

$$J_{1,2}(\rho) = \frac{1 - \sqrt{1 - 4p\rho(1 - \rho)}}{2}, \quad (5.147)$$

which is expected since the flow should equal the result for even total number of holes and it has to be phase independent in our process. Note that in phase 2 the results for ρ_- , ρ_+ and v are independent of β . Figure 5.6 shows the exact defect velocity $v(\rho)$ for $p = 3/4$ and two different values of β . For $\rho < 1/2$ the velocity is independent of β and the system is in phase 2. For $\rho > 1/2$ and $\beta = 3/4$ ($= p$) the system remains in phase 2 (lower curve). For $\rho > 1/2$ and $\beta = 1/2$ the system is in phase 1 (upper curve). At the critical density $\rho = 1/2$ there is a discontinuity in $dv/d\rho$ indicating a first-order transition. This is expected, since the model for random-sequential dynamics shows the same type of transition.

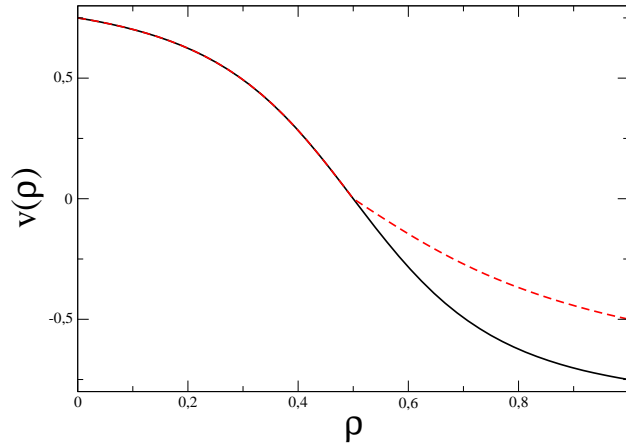


Figure 5.6: Exact defect velocity for $p = 3/4$ and two values of β . The regime $\rho \leq 1/2$ is independent of β . For $\rho > 1/2$: upper curve $\beta = 1/2$ and lower curve $\beta = 3/4$.

Now one can calculate the line on which the defect velocity vanishes (shown in figure 5.1), which is given by

$$\rho(p, \beta)|_{v=0} = \begin{cases} \frac{p-\beta}{p-\beta^2}, & \text{for } 0 \leq \beta \leq 1 - \sqrt{1-p}, \\ 1/2, & \text{for } 1 - \sqrt{1-p} \leq \beta \leq 1. \end{cases} \quad (5.148)$$

5.5.5 Density profiles

The excess hole changes the density profile. We have computed some density profiles numerically. Figure 5.7 shows density profiles $d(i)$ in the frame of the defect' (excess hole plus particle to its right). The variable i counts the distance to the right of the defect. One sees an algebraic typical for phase 2. Using the corresponding formula for the model with open boundaries ($d(i) - 1/2 \sim 1/\sqrt{i}$ to leading order in n) yields a good agreement with simulation data. In front of the defect one sees oscillations breaking the left-right symmetry. We expect the profiles in phase two to be limiting shock profiles with equal densities to the left and right. This is of interest since the ASEP has numerically be used to study shocks in lattice fluids with parallel dynamics [41].

Figure 5.8 shows different density profiles for $\rho = 1/2$ and $p = 9/10$. The continuous curve is typical for phase 1. At the boundaries there are oscillations around the value of ρ that

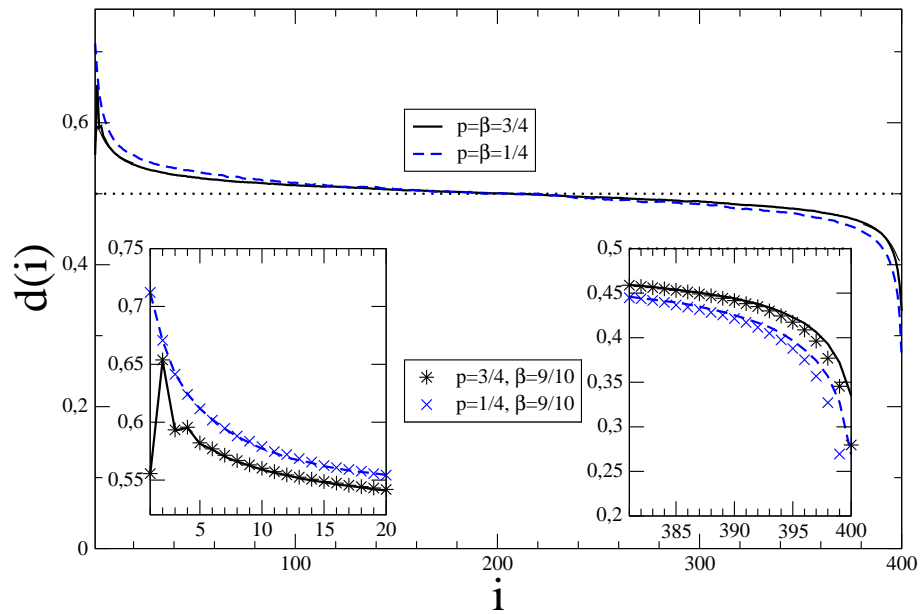


Figure 5.7: Density profiles for $\rho = 1/2$ and $p = \beta$. The system is in phase 2. The form of the profile shows an algebraic decay as in the model with open boundaries. In the windows the influence of changing β are shown.

decay exponentially fast, so that the overall density is realized in the bulk. The other two curves correspond to the regions $1 - \sqrt{1-p} < \beta < p$ and $\beta > p$ in phase 2, see diagram 5.1.

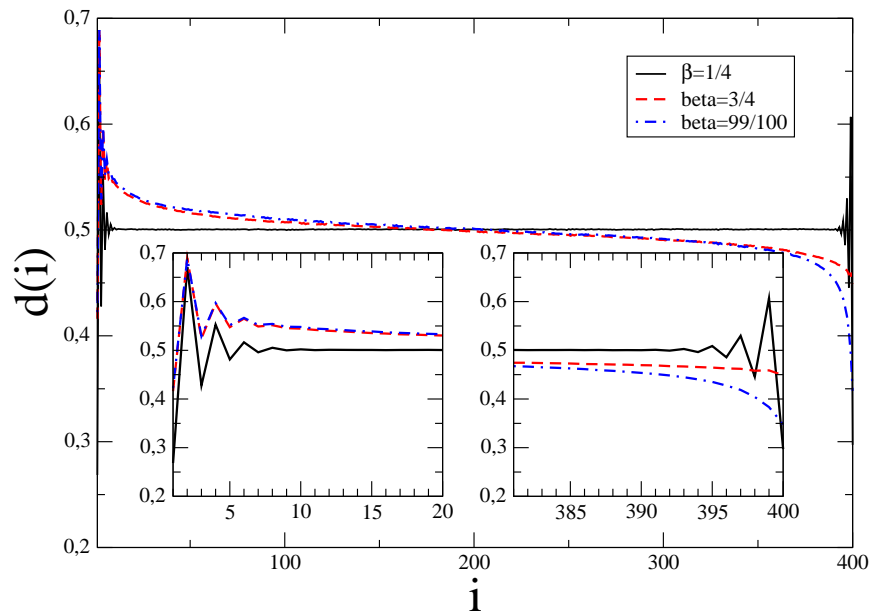


Figure 5.8: Density profile for $\rho = 1/2$ and $p = 9/10$. The continuous curve shows the profile for phase 1. The other curves correspond to phase 2.

5.6 The partially deterministic case $p = 1$

Here the matrix-product as given before holds only for $\rho \geq 1/2$. For $\rho < 1/2$ there can be several odd gaps between the particles, depending on the initial condition and the state is factorizable. The probability to find two neighboring particles is zero. All other configurations are equally probable. For $\rho > 1/2$ the state is of the form (5.16) with (2.40-2.50) and (5.17). However now it is natural to take $t(00) = 0$, since no neighboring hole pairs occur. Then choosing D and E according to (4.3) leads to the algebra

$$D_1 E_1 = D_1 + \mathbb{1}, \quad \langle W_1 | E_1 D_1 = \langle W_1 | (D_1 + \mathbb{1}), \quad D_1 | V_2 \rangle = \frac{1-\beta}{\beta} | V_2 \rangle. \quad (5.149)$$

The representation given before obviously no longer holds. Here we can choose two-dimensional commuting matrices:

$$D_1 = \begin{pmatrix} 0 & \frac{1-\beta}{\beta} \\ \frac{1-\beta}{\beta} & 0 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 1 & \frac{\beta}{1-\beta} \\ \frac{\beta}{1-\beta} & 1 \end{pmatrix}, \quad (5.150)$$

with $|V\rangle = ((1-\beta, 1-\beta), (1, 1))^t$ and for example $\langle W| = ((0, 0), (1, 1))$. For p approaching 1 one sees that the formula for the critical density (5.129) yields $\rho_c = \beta/(1+\beta)$ which can not increase $1/2$. For higher values of ρ the system is purely in phase 1. The formula for the occupation behind the defect becomes for example $\rho_- = (2\rho - 1)/(\rho - \beta(1 - \rho))$. The density profile $d(i)$ is oscillating because of the strong particle-hole attraction [24]. For $\rho = 1/2$ it oscillates between two purely linear profiles ($1 - x$ and x in terms of a scaling function $x(i) = i/[2(N - 1)]$) for the even and the odd sub-lattice, see figure 5.10 a. Adding exactly one particle to the system at density $1/2$ or equivalently removing a hole pair as in figure 5.10 b leads to symmetric (probably quadratic) solutions for the two sublattices. The functional parity then alternates with the number of particles added. For high values of ρ these oscillations are damped out and remain only near the defect. However the numerics in figure 5.9 shows lots of noise for the highly correlated case $p = 1$. We finally note that the purely deterministic case $p = \beta = 1$ has been considered as a high-speed cellular automaton for traffic [79]

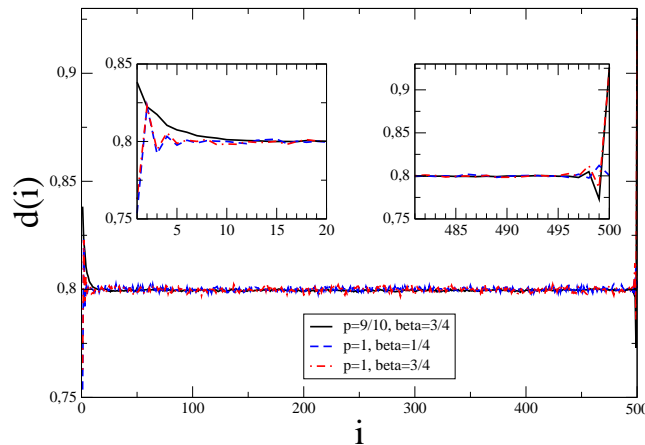


Figure 5.9: Density profiles for $\rho = 4/5$ for high values of p in comparison.

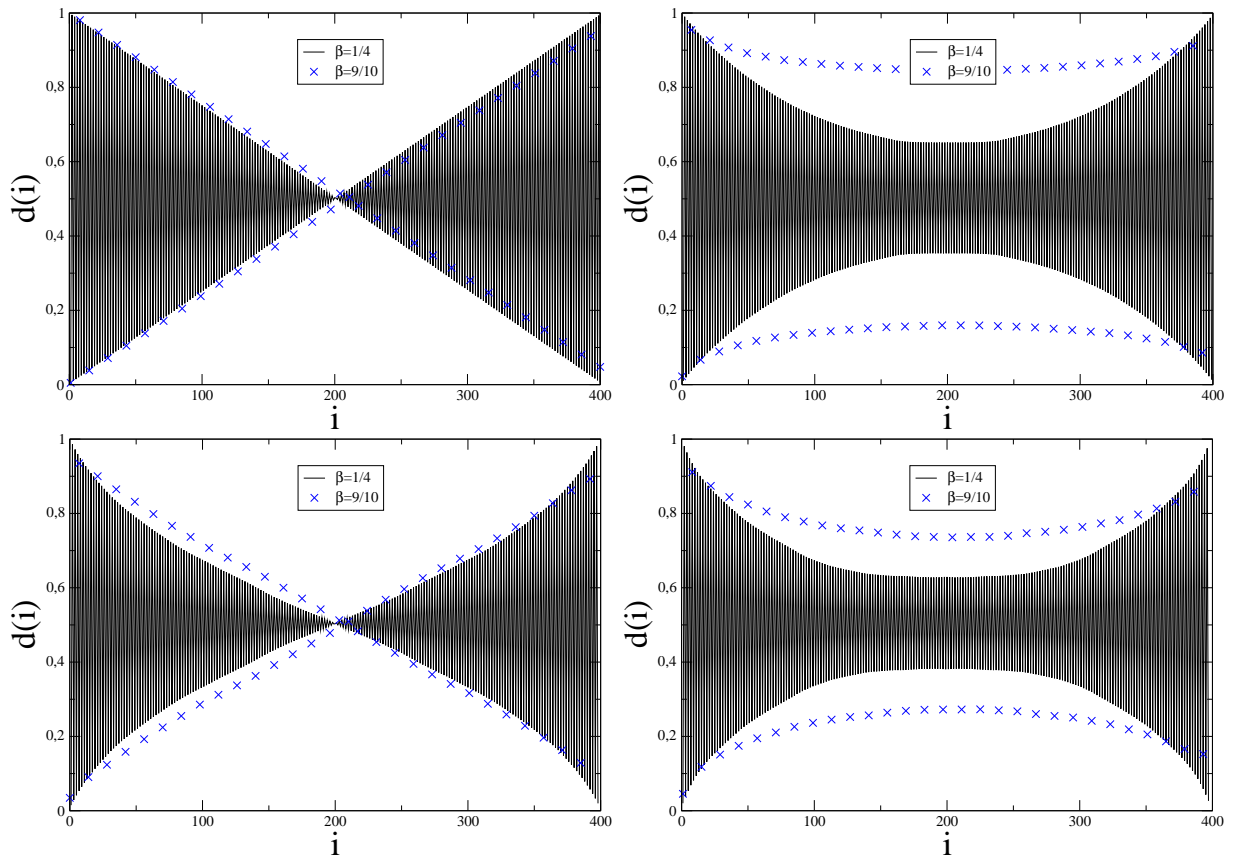


Figure 5.10: Density profiles for $p = 1$ and two values of β in comparison. The system size is a) $L = 602$, b) $L = 600$, c) $L = 598$ and d) $L = 596$ for always $N = 201$ particles so that in a) the density is exactly $\rho = 1/2$ and then increased stepwise.

5.7 Outlook

To obtain the exact expressions for asymptotic quantities we derived the generating function of the grand-canonical normalization. It would be interesting to know the exact expression of the canonical function (where particle and site number are fixed). Already the simpler second-class particle case $p = \beta$ is of interest; here most of the known functions can be expressed in terms of Narayana numbers [80]. Once knowing the canonical normalization one can close a gap of analogies to equilibrium lattice walks established recently [81–83]. Using the representation (5.37) one can derive a recursion formula similarly as in [23]. Although the recursion is more difficult one might be able to work it out. The calculation of the density profiles is of large interest for further investigations. Also the case $p = 1$ for which some results were mentioned shows obviously a non-trivial behavior. The analytic form of the density profiles is still unknown. A first step could be a diagonalizable matrix representation for the C matrix. Further it would be interesting to study other processes under parallel dynamics with defect dynamics. Note that for parallel update the ASEP with a single defect has not a natural equivalence, since the evolution of configurations in which the pattern 120 occur are not well-defined: under parallel dynamics 1 and 2 can not move to the right at the same time. However the process considered in this chapter solves this situation. 120 corresponds to 10100. This moves into 01100 (210) at

rate $\beta(1-p)$ (12 exchange), into 10001 (102) at rate $p(1-\beta)$ (20 exchange) and into 01001 (201) at rate $p\beta$ (12 exchange, then 10 exchange). However other definitions involving a different probability for particles of the second species to move may be considered. Here one can possibly make use of [84], where it is shown which local non-equilibrium dynamics lead to the same stationary state. Several of these aspects are in preparation [85].

6 The ASEP with maximum velocity two

Here we consider the slightly more general pendant to the process investigated exactly in chapter 3. The system evolves in continuous time with respect to the following rules:

$$\begin{aligned} 100 &\rightarrow 010, & \text{with rate } p_1, \\ &\rightarrow 001, & \text{with rate } p_2, \\ 101 &\rightarrow 011, & \text{with rate } \beta. \end{aligned}$$

One might think of one-dimensional highway traffic or the movement of molecular motors along a filament for example. The focus of interest is on cases in which the steady state of the system can be solved exactly. It will turn out that this is difficult for general choice of the rates. However for special choices the steady state becomes a factorized form or a matrix product form and for the case of only two particles the exact solution can be obtained for arbitrary rates.

6.1 Exactly known cases

Although a full exact description of this model is still missing some simple cases are already known:

- The case $p_1 = \beta$
This case was studied in [72] and turned out to have a uniform stationary state: all configurations with fixed system size and particle number have the same weight.
- The case $p_2 = 0$
The weights (for $\beta > 0$) are of the two-cluster form [72, 86]:

$$P(\tau_1, \tau_2, \dots, \tau_L) = \prod_{i=1}^L t(\tau_i, \tau_{i+1}), \quad (6.1)$$

with some simple two-site factors $t(\tau_i, \tau_{i+1})$.

6.2 Mapping onto a Mass-transport model

For analytical investigations it is more appropriate to represent the steady state of the model through inter-particle distances, i.e. to write the probability for a configuration as $P(m_1, m_2, \dots, m_N)$. This formally corresponds to a mapping onto a model in which occupied sites are replaced by sites and blocks of unoccupied sites to the right of such occupied site by ‘masses’ (units of particles) staying on the same site. Thus the new model comprises N sites and $M := L - N$ particles. In this model one or two particles may leave a certain site to the left with the following rates:

$$\gamma(l|m) = \begin{cases} p_1, & \text{for } l = 1, m > 1, \\ p_2, & \text{for } l = 2, m > 1, \\ \beta, & \text{for } l = 1, m = 1. \end{cases} \quad (6.2)$$

This is a special case of the class of ‘mass-transport models’, see section 2.3. Using the condition (2.77) one can reproduce the two known cases mentioned above.

6.3 Two-particle sector

Consider only two particles on a ring. We found that the un-normalized steady-state weights $f(m, n)$ obey a recursion that relates the weights for system size L to corresponding weights for system sizes $L - 1$ and $L - 2$ in the following manner:

$$f(m, n) = \omega_L(p_1)f(m-1, n) + p_2f(m-2, n), \text{ for } m \geq n \text{ and } m+n = L-2 \geq 3, \quad (6.3)$$

with the piecewise defined function

$$\omega_L(p_1) = \begin{cases} p_1, & \text{for } L \text{ even,} \\ 1, & \text{for } L \text{ odd.} \end{cases} \quad (6.4)$$

It is convenient to take $p_1 = 1$ to get rid of the even/odd dependence of the lattice size at this stage. The case $p_1 = 0$ was already studied extensively in chapter 2. Concluding we have from now on $\omega_L(p_1 = 1) \equiv 1$. We note that the process with two particles can be interpreted as a random walk with two reflecting boundaries which, however, has to our knowledge not been studied before [87, 88]. In terms of the functions

$$y_n := \left(\frac{1 + \sqrt{1 + 4p_2}}{2} \right)^n + \left(\frac{1 - \sqrt{1 + 4p_2}}{2} \right)^n = \frac{1}{2^{n-1}} \sum_k \binom{n}{2k} (1 + 4p_2)^k \quad (6.5)$$

the solution of (6.3) (for $m \geq n$ and $m \geq 1$) is

$$f(m, n) = \frac{\beta y_{m+n} + (2p_2 + 1 - \beta)y_{m+n-1}}{1 + 4p_2} + (-1)^{n-1} \frac{(1 - \beta)p_2^n y_{m-n}}{1 + 4p_2}. \quad (6.6)$$

Let us now interpret this result. One sees that the first term only depends on $n + m$ and therefore it is constant for given system size L . The second term with its pre-factor $(-1)^{n-1}$ indicates that in general there are oscillations. Thus the weights $f(m, n)$ depend on the parity of n . One also sees that there are no oscillations in the known cases $\beta = 1$ and $p_2 = 0$.

To relate the weights $f(m, n)$ to probabilities one has to calculate the normalization by summing over (6.6). After some algebra this gives for $L \geq 3$:

$$\begin{aligned} Z_{L,2} &= \sum_{m=0}^{L-2} f(m, L-2-m) \\ &= \left[\frac{\beta(L-1)}{1 + 4p_2} - \frac{2(1-\beta)(2p_2+1)}{(1+4p_2)^2} \right] y_{L-2} + \left[\frac{(2p_2+1-\beta)(L-1)}{1 + 4p_2} - \frac{2p_2(1-\beta)}{(1+4p_2)^2} \right] y_{L-3}. \end{aligned} \quad (6.7)$$

Then the probability for a certain configuration is given by $P(m, n) = Z_{m+n+2,2} f(m, n)$.

This enables for example to calculate the velocity:

$$v(\beta, p_2) = \beta P(1, L-3) + (1 + 2p_2) [1 - P(1, L-3) - P(0, L-2)]. \quad (6.8)$$

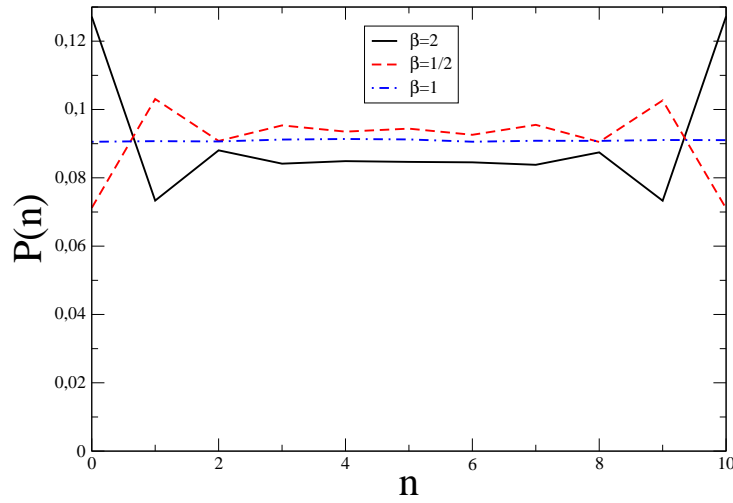


Figure 6.1: Comparison of the gap probabilities in the two-particle sector for $L = 12$ for $p_1 = 1$ and different values of β which is taken here as the inverse $\beta = p_2^{-1}$.

6.3.1 Some simple limits

- $\beta = 1$

In this case $f(m, n)$ becomes a function of $m + n$ only. Thus all weights for a given system size $L = m + n + 2$ are the same which is in accordance with the known results.

- $p_2 = 0$

In this case one has $y_n(0) = 1$ which gives

$$f(m, n) = \begin{cases} \beta, & \text{for } n = 0, \\ 1, & \text{for } n \geq 1. \end{cases} \quad (6.9)$$

Thus the corresponding mass-transport model has a factorized steady state with single site weights $s(0) = \beta$ and $s(m) = 1$, for $m \geq 1$. This corresponds to the known two-cluster form and simple matrix form discussed in a previous section.

- $p_2 = 1$

This case (later referred to as ‘Fibonacci Case’) is interesting and has to our knowledge not been investigated before. We will treat it in detail in the next section.

6.3.2 The ‘Fibonacci case’ $p_1 = p_2 = 1$

Taking additionally $p_2 = 1$, formula (6.6) simplifies. We get the following nice recursion for the un-normalized weights $f(m, n)$ of an arbitrary configuration:

$$f(m, n) = F_{m+1}F_{n+1} - (2 - \beta)F_{m-1}F_{n-1}, \text{ for } m, n \geq 0 \text{ and } m + n \geq 1. \quad (6.10)$$

Here F_k denotes the k th Fibonacci number satisfying $F_{k+2} = F_{k+1} + F_k$ given explicitly by

$$F_k = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^k - \left(\frac{1 - \sqrt{5}}{2} \right)^k \right], \text{ for } k \geq 0, \quad (6.11)$$

and we define $F_{-1} := 1$ so that $F_{-1} + F_0 = F_1$. For this case the calculation of the normalization is easier than in the general case. With

$$Y_L := \sum_{n=0}^{\infty} (n+1) \binom{n}{L-4-n}, \quad (6.12)$$

the normalization can be rewritten as

$$Z_{L,2}(\beta) = Y_{L+2} - (2 - \beta) [Y_{L-2} + 2F_{L-3}], \text{ for } L \geq 3. \quad (6.13)$$

Some first numerical values are:

$$Z_3 = 2, \quad Z_4 = 1 + 2\beta, \quad Z_5 = 6 + 2\beta, \quad Z_6 = 10 + 5\beta, \quad Z_7 = 22 + 8\beta, \dots \quad (6.14)$$

On regarding (6.10) one can directly identify two simple cases:

- $\beta = 1$
The weights reduce to $f(m, n) = F_{m+1}F_{n+1} - F_{m-1}F_{n-1} = F_{m+n}$ and again the uniform measure is recovered.
- $\beta = 2$
In this case the corresponding mass-transport model on two sites (rather than two particles) has a factorized steady state. Thus we have one factor for each mass. This can also be understood since the average outflow of each non-vanishing mass is constant. The single-site weights become Fibonacci numbers $s(n) = F_{n+1}$. However a look on systems with more than 2 particles immediately shows that this factorization no longer holds.

Figure (6.2) shows for $L = 12$ the dependence of the gap probabilities on the value of β . The uniform case $\beta = 1$ corresponds to the flat line $P(m) = 1/11$ for $m = 0 \dots 10$. It is interesting to note that there are four knots which is interesting especially for $n = 2$: The probability to have two empty sites in front is (almost) independent of β in this case. The bottom figure shows the same curves for twice the system size ($L = 24$). There we displayed only three curves for a clearer view. Here one finds six knots. For large systems the oscillations are damped rapidly with increasing n ; the two particles do no longer see each other and behave like two non-interacting random walkers. It turns out that in the Fibonacci case ($p_1 = p_2 = 1$) the recursion relation (6.3) becomes

$$f(m, n) = f(m-1, n) + f(m-2, n), \text{ for } m \geq 2. \quad (6.15)$$

Note that in this case one does not have explicitly $m \geq n$. This now is a recursion relation that can be expressed as a matrix state

$$f(m, n) = \text{tr} (DE^m DE^n) \quad (6.16)$$

in the following form:

$$DEE = DE + D. \quad (6.17)$$

Which is a new sort of diffusion algebra [89] compared to the usual DEHP algebra $DE = D + E$. A representation for this ‘algebra’ is easily obtained. Introduce the Q-matrix

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad (6.18)$$

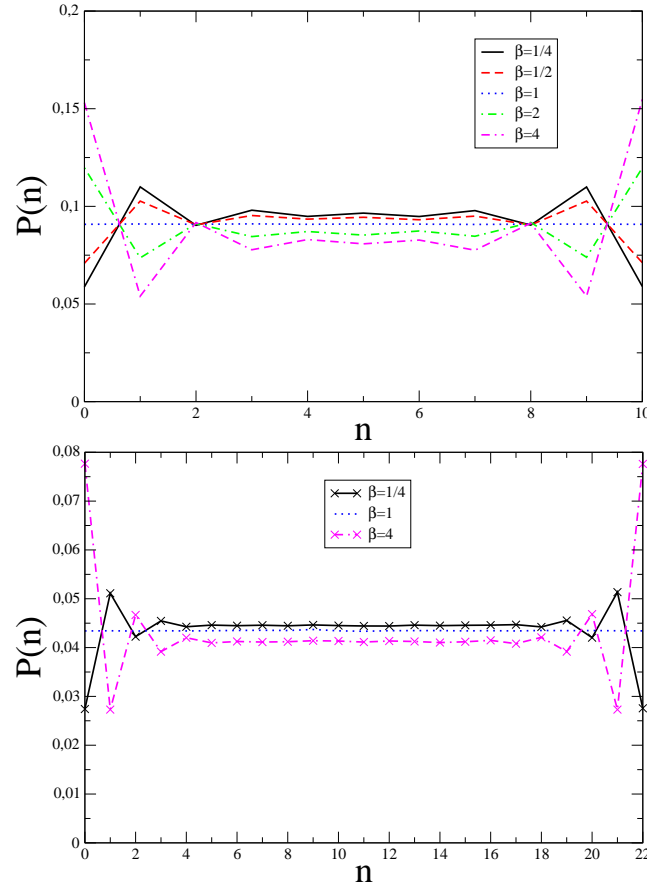


Figure 6.2: Comparison of the gap probabilities in the two-particle sector for the Fibonacci case $p_1 = p_2 = 1$. In the top diagram one has $L = 12$ and in the bottom diagram $L = 24$.

which fulfills

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}, \quad n \geq 0 \quad (6.19)$$

and define the following matrices:

$$D_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 \\ 0 & i \cdot \sqrt{2 - \beta} \end{pmatrix}, \quad (6.20)$$

with i the imaginary unit. Then the matrices E and D have an effectively four-dimensional representation:

$$E = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}, \quad D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}. \quad (6.21)$$

However the recursion (6.17) for this special process no longer holds for more than two particles on the ring and therewith the exact solution of the Fibonacci case for arbitrary number of particles remains an (interesting) open problem.

6.3.3 More than two particles

Obviously for more than two particles on the ring the headway distribution can no longer be symmetric. It decays with increasing headway. However the oscillations remain. Figure 6.3 shows the distributions for $N = 2, 3$ and 4 on a small lattice of $L = 12$ sites.

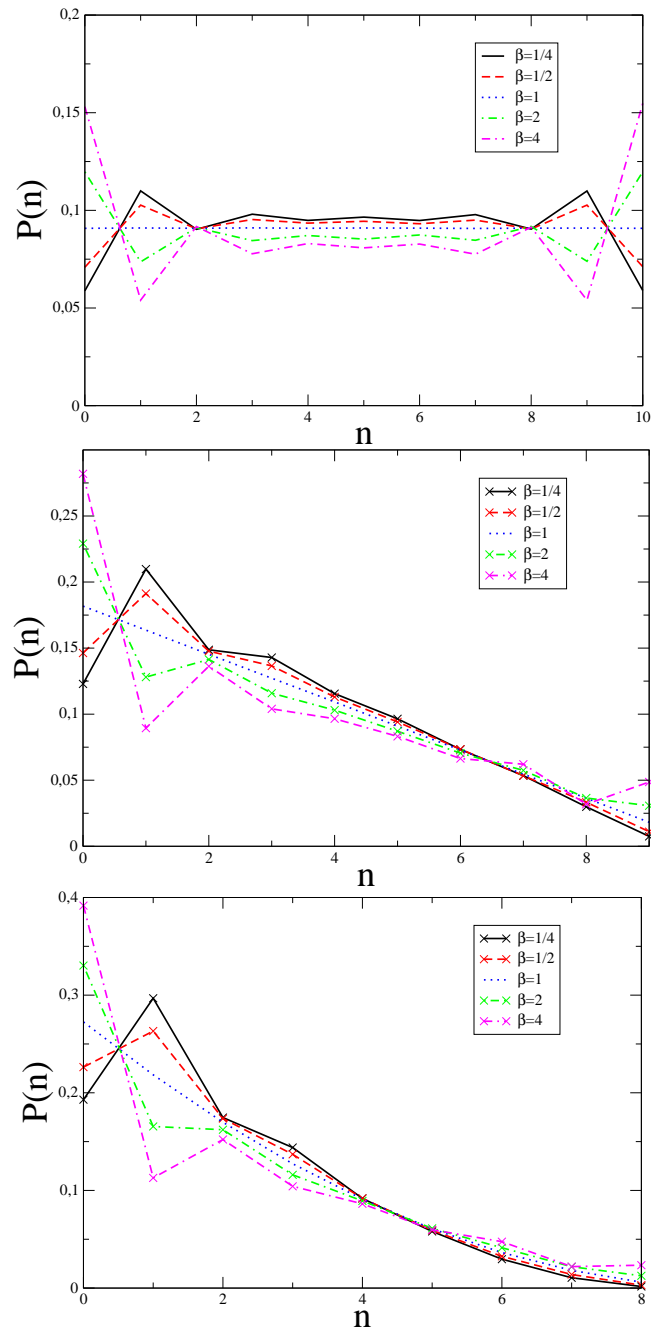


Figure 6.3: Comparison of the gap probabilities in for 2, 3 and 4 particles (from top to bottom) for $L = 12$ for the Fibonacci case $p_1 = p_2 = 1$ for different values of β from a computer simulation.

6.4 Mean-field theory

A so-called mean-field theory neglects certain correlations and is often used in statistical mechanics. We perform such a theory for the corresponding mass-transport model.

Neglecting correlations between adjacent masses leads to

$$\begin{aligned} (c + p_2 s)P(0) &= \beta P(1) + p_2 P(2), \\ (c + p_2 s + \beta)P(1) &= cP(0) + p_1 P(2) + p_2 P(3), \\ (c + p_2 s + p_1 + p_2)P(m) &= p_2 s P(m-2) + cP(m-1) + p_1 P(m+1) + p_2 P(m+2), \\ &\text{for } m \geq 2, \end{aligned} \quad (6.22)$$

with the short-hand notations

$$s := 1 - P(0) - P(1) \text{ and } c := \beta P(1) + p_1 s. \quad (6.23)$$

Note that in the ASEP picture this is an improved mean-field theory that takes into account the distance to the next particle ahead. In this context with parallel update the theory was referred to as car-oriented mean-field theory (COMF) [90]. We now introduce the generating function

$$Q(z) = \sum_{m=0}^{\infty} P(m) z^m. \quad (6.24)$$

Summing up the equations for $P(m)z^m$ for $m = 0 \dots \infty$ leads to a rational expression for $Q(z)$. The numerator is a polynomial of degree 3 and the denominator is a polynomial of degree 4. It turns out that the denominator has a singularity at $z = 1$. However this is removable since the numerator itself has a zero at $z = 1$. One obtains

$$Q(z) = \frac{(\beta - p_1 - p_2)P(1)z^2 - wz - p_2 P(0)}{p_2 s z^3 + (c + p_2 s)z^2 - (p_1 + p_2)z - p_2}, \quad (6.25)$$

with $w := (p_1 + p_2)P(0) + p_2 P(1)$. A useful check of this equation is $Q(0) = P(0)$ and $Q(1) = 1$. The density ρ in the corresponding asymmetric exclusion process is

$$\partial_z(zQ(z))|_{z=1} = \sum_{m=0}^{\infty} (m+1)P(m) = \rho^{-1}. \quad (6.26)$$

This gives $P(1)$ in terms of $P(0)$ and ρ :

$$P(1) = \frac{[2p_2(1 + \rho) + p_1] P(0) - (4p_2 + p_1)\rho}{(\beta - p_1)(1 - \rho) - 2p_2}. \quad (6.27)$$

The remaining probabilities can for example be calculated by evaluating the integral

$$P(m) = \frac{1}{2\pi i} \int_{\mathcal{C}_0} \frac{Q(z)}{z^{m+1}} dz \quad (6.28)$$

over the contour \mathcal{C}_0 encircling the origin or directly from $Q(z)$. The flow-density relation is $J(\rho) = \rho(c + 2p_2 s)$.

However at this stage already one equation is missing. One needs an additional relation between $P(1)$ and $P(0)$ to be able to express everything in terms of the density only. In fact the missing relation can be extracted from the generating function. The numerator of $Q(z)$ can be rewritten in terms of its zeros z_0^\pm as $(\beta - p_1 - p_2)P(1)(z - z_0^+)(z - z_0^-)$. The singularity in the unit circle has to be removed by one of these factors for $Q(z)$ to be analytic. A similar problem occurred also for the Nagel-Schreckenberg model with $v_{max} = 2$ in the COMF approximation [90]. This yields the missing relation between $P(1)$ and $P(0)$. The *first relation* is $p_2 = 0$. Thus for maximum velocity 1, no additional relation is needed. This is the known case studied in [86]. Figure (6.4) shows the known fundamental diagram for $p_2 = 0$.

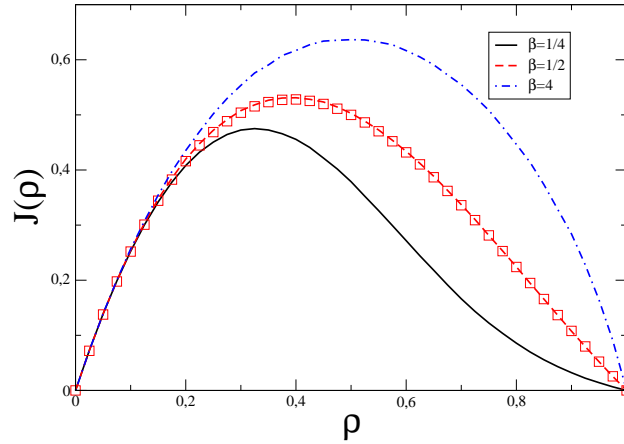


Figure 6.4: Fundamental diagrams for the Fibonacci case $p_1 = p_2 = 1$ for different values of β from computer simulations with $L = 200$. The squares show the known exact result for the uniform case $p_1 = \beta$ in the thermodynamic limit.

The *second relation* leads to expressions for $P(m)$ that are independent of ρ for all $m \geq 0$ and therefore are un-physical. The *third relation*, which is the interesting one, can be written as

$$p_2 = (p_1 - \beta) \frac{P(1)[1 - P(0)]}{P(1) - P(0)[1 - P(0)]} \frac{cP(0) - \beta P(1)}{cP(1) - p_1 sP(0)}. \quad (6.29)$$

With the help of this equation and keeping in mind that $Q(z)$ has to be analytic in the unit circle everything can be calculated straightforwardly.

Now consider $p_2 > 0$. Inserting (6.29) into $Q(z)$ gives the final expression for the generating function

$$Q(z) = \frac{aP(0) + bz}{a + [p_1 sP(0) - cP(1)]z + [\beta P(1) - cP(0)][1 - P(0)]z^2}, \quad (6.30)$$

where we defined

$$a = (p_1 - \beta)P(1)[1 - P(0)], \quad (6.31)$$

$$b = (p_1 - \beta)P(1)^2 - p_1 P(0)(P(1) - P(0)[1 - P(0)]). \quad (6.32)$$

Now one can set $p_2 = 1$ without loss of generality. If $p_1 = \beta$ then the factor $(p_1 - \beta) = 0$ on the right-hand side has to be canceled by a corresponding factor in the denominator. In fact, in this case both factors in the denominator are the same, leading to $P(1) = P(0)[1 - P(0)]$.

Consider now $\beta = p_1 + p_2$ which is an interesting line that we pointed out in [31]. The rate at which a particle changes its given headway is headway-independent. The condition is weaker than the condition for a factorized state (every configuration is equally probable if $p_1 = \beta$). Substituting this into (6.25) one sees that the coefficient of z^2 vanishes. Demanding that the denominator has the same zero as the numerator gives the missing relation:

$$p_2 P(1)^2 = p_1 P(0)(P(0)[1 - P(0)] - P(1)). \quad (6.33)$$

For $Q(z)$ one gets the simple expression

$$Q(z) = \frac{pP(0)A^2}{pA^2 - pAz - z^2}, \quad (6.34)$$

with $A := P(0)/P(1)$ and $p := p_1/p_2$, which can nicely be expanded to obtain $P(m)$. Figure 6.5 shows the mean-field distribution for two different choices of parameters in comparison with computer simulations. While $P(n)$ decays rapidly, one sees the characteristic even/odd oscillations which are well reproduced by mean field.

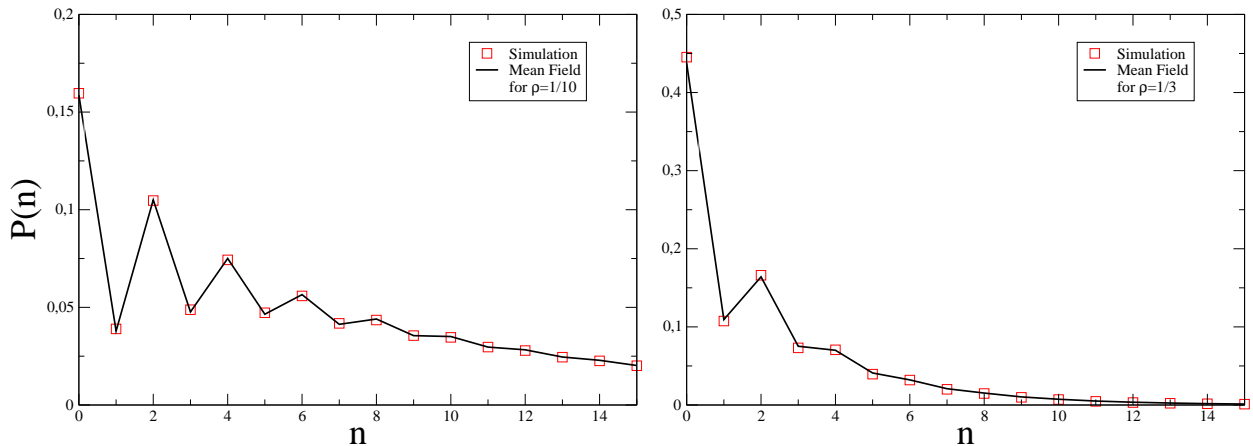


Figure 6.5: Headway distribution $P(n)$ from a computer simulation with $L = 1000$ in comparison with mean field for $p_2 = 1$. Left: $p_1 = 0.1$, $\beta = 1.1$ and $\rho = 0.1$. Right: $p_1 = 0.2$, $\beta = 1.2$ and $\rho = 1/3$.

6.4.1 The Fibonacci case

To $p_1 = p_2 = 1$ we previously referred to as Fibonacci case, since the two particle weights are related to the Fibonacci numbers. Especially for $\beta = 2$ the two-particle state is factorizable with single-site weights $s(n) = F_{n+1}$. Here (6.34) becomes with the help of (6.33):

$$Q(z)|_{p_1=p_2=1, \beta=2} = \frac{P(0)A^2}{A^2 - Az - z^2}, \quad \text{with } A = P(0)/P(1). \quad (6.35)$$

Then the probabilities $P(m)$ are given by the Fibonacci numbers:

$$P(m) = P(0) \left(\frac{P(1)}{P(0)} \right)^m F_{m+1}. \quad (6.36)$$

Here one obtains from (6.33): $P(1)/P(0) = (\sqrt{5 - 4P(0)} - 1)/2$ and relating $P(0)$ to the density gives

$$P(0) = \frac{\rho}{(1+\rho)^2} \frac{5 + 4\rho - \sqrt{5 - 4\rho^2}}{2}, \quad (6.37)$$

$$P(1) = \frac{\rho^2}{(1+\rho)^2} - \frac{5 + 4\rho - (3 + 2\rho)\sqrt{5 - 4\rho^2}}{2}. \quad (6.38)$$

Figure 6.6 shows the gap probabilities for the Fibonacci case.

Note that this result is also obtained from (6.29) and (6.25).

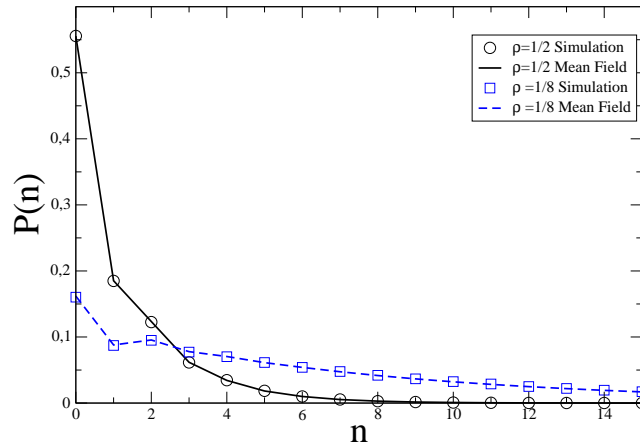


Figure 6.6: Headway distribution $P(n)$ from mean-field for the Fibonacci case $p_1 = p_2 = 1$ and $\beta = 2$ for $\rho = 1/8$ and $1/2$ compared with computer simulations.

6.4.2 The ASEP with excess-mass formation

In the factorized form $(\beta - p_1 - p_2)P(1)(z - z_0^+)(z - z_0^-)$ of the numerator of $Q(z)$ in (6.25) one sees that also the case $P(1) = 0$ is special. $P(1) = 0$ has the consequence that $P(2n + 1)$ vanishes generally. So this process is realized on the even sublattice in mean field. This violation of symmetry is a direct consequence of the hopping parameters and does not depend on the initial configuration. Considering (6.29), the case $P(1) = 0$ can be realized only with $p_1 s P(0) = 0$ which leads to $p_1 = 0$ and β , arbitrary. So in general $p_1 = 0$ and $p_2 = 1$ leads here to

$$P(0) = \frac{2\rho}{1 + \rho}. \quad (6.39)$$

The probabilities $[P(n)](\rho)$ for a particle being followed by n sites become for $n = 0, 1, \dots$:

$$P(2n) = P(0) [1 - P(0)]^n, \quad (6.40)$$

$$P(2n + 1) = 0. \quad (6.41)$$

The flow simply reads

$$J(\rho) = 2\rho [1 - P(0)] = \frac{2\rho(1 - \rho)}{1 + \rho}. \quad (6.42)$$

Thus the mean-field theory correctly predicts the thermodynamic limit. This is in contrast with the factorization condition for mass-transport models in its original form (2.77). For the thermodynamically large system the single excess hole plays no role for the overall headway distribution of particles. However note that the steady state has no product measure form though.

6.5 Parallel dynamics

The generalization to arbitrary hop probabilities for parallel dynamics

$$\begin{aligned} 100 &\rightarrow 100, && \text{with probability } p_0, \\ &\rightarrow 010, && \text{with probability } p_1, \\ &\rightarrow 001, && \text{with probability } p_2, \\ 101 &\rightarrow 101, && \text{with probability } \beta_0, \\ &\rightarrow 011, && \text{with probability } \beta_1, \end{aligned}$$

with $p_0 + p_1 + p_2 = \beta_0 + \beta_1 = 1$. As mentioned before this is precisely the traffic model discussed in [71]. There it was argued from an analogy to the chipping model [91] that this process can not lead to a condensation transition with a global phase separation in the system. Some aspects with focus on partially deterministic dynamics have been considered in [92]. It is further related to the Fukui-Ishibashi traffic model [93–95] for which some stationary properties could be calculated [96]. For the case $p_1 = 0$ it was shown before, that the system qualitatively shows the same behavior although the calculations become more difficult due to the attraction of particles and hole-pairs. The factorization criterion of Evans, Majumdar and Zia [64, 65] becomes here

$$\frac{p_1}{p_0} = \frac{\beta_1}{\beta_0}. \quad (6.43)$$

Thus for this choice of parameters the steady state is factorizable. The condition can be interpreted as follows: The quantity p_1/p_0 is the rate (!) at which a particle moves a single site if it has at least two empty sites in front. This has to equal the rate β_1/β_0 at which a particle moves a single site if it has only one empty site in front. Therewith the factorization condition is the same as for random update. We note that this can be generalized to higher maximum velocities $v_{\max} > 2$). For general v_{\max} the condition for a factorized steady state is always that the rate at which a particle moves a certain number of sites is independent of the headway. For further investigations one could perform a mean-field theory also for the parallel case. The COMF equations are given in the appendix.

7 Related processes

In the following some processes are presented that can be obtained from the ASEP with excess-mass formation either by a direct mapping or are closely related to it. At first we present a new zero-range process with two particle species that can be obtained by a mapping. Afterwards we shortly mention a related $v_{max} = 2$ -traffic model with uniform state. Then, in 7.3 we derive a quite general mass-transport model that contains the ASEP with excess-mass formation as special case. Finally we show a generalized ASEP where sites can be occupied also by two particles which exhibits excess-mass formation and show its relation to the defect ASEP.

7.1 A corresponding zero-range process with two particle species

Imagine a zero-range process in which particles hop to the left. In the system should be exactly one second-class particle. In each infinitesimal time-step only one particle can move. Note that there is no internal structure in each mass, i.e. no ordering of particles belonging to species 1 and 2.

The mass-transport model defined by (3.1-3.2) can be obtained by representing each 1-species particle by two 1-species particles and the 2-species particle by a single 1-species particle.

The probability for a particle to move depends on the number of 1-particles on the site and on the presence of a 2-particle and can be denoted by $u(n, \tau)$ for the movement of 1-species particles and $v(\tau, n)$ for the movement of a 2-species particle. n denotes the number of 1-particles on the site and τ (which can take values 0 or 1) denotes the presence of a 2-particle on the site.

Take the simple probabilities

$$u(n, \tau) = 1, \quad \text{for } n \geq 1, \quad (7.1)$$

$$v(\tau, n) = \beta [1 - \theta(n)], \quad \text{for } \tau = 1. \quad (7.2)$$

Of course it is $u(0, \tau) = u(0, n) = 0$. In words this formula means that 1-particles hop unaffected by the presence of 2-particles with constant rate 1. A 2-particle in the presence of 1-particles can not move but if a 2-particle is alone on a site then it can move with rate β .

According to the mass-transport model the process obeys the algebra

$$\mathcal{G}_i \mathcal{U}_0 = \mathcal{U}_i, \quad (7.3)$$

$$\mathcal{U}_i \mathcal{G}_{j+1} - \mathcal{U}_i \mathcal{G}_j = 0, \quad (7.4)$$

$$\mathcal{G}_i \mathcal{G}_{j+1} - \mathcal{G}_i \mathcal{G}_j = \beta \mathcal{G}_{i+j+1}, \quad (7.5)$$

$$\mathcal{G}_i \mathcal{U}_{j+1} - \mathcal{G}_i \mathcal{U}_j = \beta \mathcal{U}_{i+j+1}, \quad \text{for } i, j \geq 0. \quad (7.6)$$

Here we used a slightly different notation as in the mass transport model. The matrix \mathcal{U}_n represents a site occupied by the second-class particle and n first-class particles. The

matrix \mathcal{G}_n represents a site occupied by no second-class and n first-class particles. The ansatz

$$\mathcal{G}_i = \beta E^i D, \quad (7.7)$$

$$\mathcal{U}_i = E^i A \quad (7.8)$$

reduces (7.3-7.6) to (2.34-2.36) which is the algebra for the two-species ASEP with $\alpha = 1$. Writing down the master equation and using the canceling mechanism for this process leads to

$$\bar{\mathcal{G}}_0 \mathcal{U}_0 - \mathcal{G}_0 \bar{\mathcal{U}}_0 = -\beta \mathcal{G}_0 \mathcal{U}_0, \quad (7.9)$$

$$\bar{\mathcal{G}}_0 \mathcal{U}_{i+1} - \mathcal{G}_0 \bar{\mathcal{U}}_{i+1} = -\mathcal{G}_0 \mathcal{U}_{i+1}, \quad (7.10)$$

$$\bar{\mathcal{G}}_{i+1} \mathcal{U}_0 - \mathcal{G}_{i+1} \bar{\mathcal{U}}_0 = -\beta \mathcal{G}_{i+1} \mathcal{U}_0 + \mathcal{G}_i \mathcal{U}_1, \quad (7.11)$$

$$\bar{\mathcal{G}}_{i+1} \mathcal{U}_{j+1} - \mathcal{G}_{i+1} \bar{\mathcal{U}}_{j+1} = \mathcal{G}_i \mathcal{U}_{j+2} - \mathcal{G}_{i+1} \mathcal{U}_{j+1}, \quad (7.12)$$

$$\bar{\mathcal{G}}_0 \mathcal{G}_0 - \mathcal{G}_0 \bar{\mathcal{G}}_0 = 0, \quad (7.13)$$

$$\bar{\mathcal{G}}_0 \mathcal{G}_{i+1} - \mathcal{G}_0 \bar{\mathcal{G}}_{i+1} = -\mathcal{G}_0 \mathcal{G}_{i+1}, \quad (7.14)$$

$$\bar{\mathcal{G}}_{i+1} \mathcal{G}_0 - \mathcal{G}_{i+1} \bar{\mathcal{G}}_0 = \mathcal{G}_i \mathcal{G}_1, \quad (7.15)$$

$$\bar{\mathcal{G}}_{i+1} \mathcal{G}_{j+1} - \mathcal{G}_{i+1} \bar{\mathcal{G}}_{j+1} = \mathcal{G}_i \mathcal{G}_{j+1} - \mathcal{G}_{i+1} \mathcal{G}_j, \quad (7.16)$$

$$\bar{\mathcal{U}}_0 \mathcal{G}_0 - \mathcal{U}_0 \bar{\mathcal{G}}_0 = \beta \mathcal{G}_0 \mathcal{U}_0, \quad (7.17)$$

$$\bar{\mathcal{U}}_0 \mathcal{G}_{i+1} - \mathcal{U}_0 \bar{\mathcal{G}}_{i+1} = -\mathcal{U}_0 \mathcal{G}_{i+1}, \quad (7.18)$$

$$\bar{\mathcal{U}}_{i+1} \mathcal{G}_0 - \mathcal{U}_{i+1} \bar{\mathcal{G}}_0 = \mathcal{U}_i \mathcal{G}_1 + \beta \mathcal{G}_{i+1} \mathcal{U}_0, \quad (7.19)$$

$$\bar{\mathcal{U}}_{i+1} \mathcal{G}_{i+1} - \mathcal{U}_{i+1} \bar{\mathcal{G}}_{i+1} = \mathcal{U}_i \mathcal{G}_{i+2} - \mathcal{U}_{i+1} \mathcal{G}_{i+1}. \quad (7.20)$$

If we take here

$$\bar{\mathcal{G}}_0 = -\beta \mathbb{I}, \quad (7.21)$$

$$\bar{\mathcal{G}}_{i+1} = \mathcal{G}_i = \beta E^i D, \quad (7.22)$$

$$\bar{\mathcal{U}}_0 = 0, \quad (7.23)$$

$$\bar{\mathcal{U}}_{i+1} = \mathcal{U}_i = E^i A, \quad (7.24)$$

then (7.9-7.20) indeed reduces to (2.34-2.36) which completes the proof of the steady state.

7.1.1 Generalization to arbitrary density of 2-species particles

A natural extension to arbitrary densities of both species of particles is the following:

$$u(n, m) = 1, \quad \text{for } n \geq 1, \quad (7.25)$$

$$v(m, n) = \beta [1 - \theta(n)], \quad \text{for } m \geq 1. \quad (7.26)$$

For this model we now write \mathcal{C}_{ij} for a site occupied by i 1-class particles and j 2-class particles. The algebra resulting from the cancelling mechanism reads:

$$\begin{aligned} \bar{\mathcal{C}}_{ij} \mathcal{C}_{kl} - \mathcal{C}_{ij} \bar{\mathcal{C}}_{kl} = & -\theta(k) \mathcal{C}_{ij} \mathcal{C}_{kl} - \beta \theta(l) [1 - \theta(k)] \mathcal{C}_{ij} \mathcal{C}_{kl} \\ & + \theta(i) \mathcal{C}_{i-1j} \mathcal{C}_{k+1l} + \beta \theta(j) [1 - \theta(k)] \mathcal{C}_{ij-1} \mathcal{C}_{kl+1}. \end{aligned} \quad (7.27)$$

It has the following solution:

$$\mathcal{C}_{i0} = \beta E^i D, \quad i \geq 0, \quad (7.28)$$

$$\mathcal{C}_{ij+1} = E^i A^{j+1}, \quad i, j \geq 0. \quad (7.29)$$

For the corresponding tagged matrices one has:

$$\bar{\mathcal{C}}_{00} = -\beta \mathbb{1}, \quad (7.30)$$

$$\bar{\mathcal{C}}_{i+10} = \mathcal{C}_{i0}, \quad (7.31)$$

$$\bar{\mathcal{C}}_{0i+1} = 0, \quad (7.32)$$

$$\bar{\mathcal{C}}_{i+1j+1} = \mathcal{C}_{ij+1}, \quad i, j \geq 0, \quad (7.33)$$

7.2 Related processes with uniform measure

We note that instead of considering the process $100 \rightarrow 001$ and $101 \rightarrow 011$ one can think of slightly modified processes that have a simpler steady state. For example the following

$$01000 \rightarrow 00010, \text{ with rate } 1 - \text{free flow}, \quad (7.34)$$

$$1100 \rightarrow 1010, \text{ with rate } \beta - \text{accelerate}, \quad (7.35)$$

$$101 \rightarrow 011, \text{ with rate } \beta - \text{close the gap}. \quad (7.36)$$

Here all configurations are equally probable. In the first line one has the free-flow rule. A car having an empty site to the left and three free sites in front is freely moving at its maximum velocity two sites per time step. In the second line the relevant car has another one to the left which means that it belongs to a jam and stood still so this rate is decreased to a value β . In the last line we assume that a car can close the gap to its car in front with the same rate.

7.3 Derivation of a mass-transport generalization

7.3.1 Matrix representation of the ASEP with zero-range hopping

In its simplest version the Zero-Range Process (ZRP) is a model on a periodic 1-dimensional lattice with N sites labelled $l = 1, \dots, N$, see section 2.3. The master equation for this process reads:

$$\sum_{i=1}^L F(m_1, m_2, \dots, m_N) u(m_i) = \sum_{i=1}^L F(\dots, m_i - 1, m_{i+1} + 1, \dots) u(m_{i+1}) \theta(m_i). \quad (7.37)$$

It is well known that such a model has a factorized steady state, i.e. the (un-normalized) weight for a configuration (m_1, m_2, \dots, m_N) can be written as

$$F(m_1, m_2, \dots, m_N) = \prod_{\nu=1}^N f(m_\nu), \quad (7.38)$$

with some single-site weight functions $f(m)$ given by

$$f(0) = 1, \quad f(m) = \prod_{i=1}^m u(i)^{-1}, \quad \text{for } m \geq 1, \quad (7.39)$$

what can directly be checked by insertion.

The ZRP can be mapped onto an asymmetric exclusion process by letting the sites of the ZRP become the particles of the ASEP and the particles of the ZRP the unoccupied sites of the ASEP. Then the ASEP comprises $L := N + M$ sites, N particles and M

unoccupied sites, referred to as ‘holes’ and we do no longer have two or more particles at a given site. So the configuration can be expressed by $(\tau_1, \tau_2, \dots, \tau_L)$, where $\tau_l = 0$ if site l is unoccupied and $\tau_l = 1$ if it is occupied by one particle. The rate $u(m)$ at which a particle moves one site to the right depends now on the number m of holes in front. The factorization (7.38) becomes a factorization of the configuration into inter-particle distances. In this picture the master equation is rewritten:

$$\sum_{i=1}^L F(\dots, 1, 0^{m_i}, \dots) u(m_i) = \sum_{i=1}^L F(\dots, 1, 0^{m_i-1}, 1, 0^{m_{i+1}+1}, \dots) u(m_{i+1}) \theta(m_i). \quad (7.40)$$

However in the ASEP it may be more natural to have a representation of the weight for a given configuration in terms of one-site factors (one factor for each site). In fact this can be done within the framework of the matrix-product ansatz: Represent each occupied site ($\tau = 1$) by a matrix d and each empty site ($\tau = 0$) by a matrix e . Then the configuration can be written as a trace over a product of these matrices:

$$F(\tau_1, \tau_2, \dots, \tau_L) = \text{tr} \prod_{l=1}^L [\tau_l d + (1 - \tau_l) e]. \quad (7.41)$$

If we make this ansatz instead of (7.38) and insert it into the master equation (7.40) we find

$$\sum_{i=1}^L \text{tr} (\dots d e^{m_i} d e^{m_{i+1}} \dots) u(m_i) = \sum_{i=1}^L \text{tr} (\dots d e^{m_i-1} d e^{m_{i+1}+1} \dots) u(m_{i+1}) \theta(m_i). \quad (7.42)$$

We now note that it is sufficient if the matrices d and e fulfill the relation

$$u(m) d e^m d = (d + e) e^{m-1} d, \quad (7.43)$$

which can alternatively be expressed as

$$f(m)^{-1} d e^m d = d^2 + \sum_{n=0}^{m-1} f(n) e^{n+1} d. \quad (7.44)$$

We show this by inserting (7.43) into (7.42) gives

$$\sum_{i=1}^L \text{tr} (\dots d e^{m_i} d e^{m_{i+1}} \dots) u(m_i) = \sum_{i=1}^L \text{tr} (\dots d e^{m_i-1} d e^{m_{i+1}+1} \dots) u(m_{i+1}) \theta(m_i). \quad (7.45)$$

Now applying (7.43) on both sides leads to

$$\begin{aligned} & \sum_{i=1}^L \text{tr} (\dots d e^{m_i-1} d e^{m_i} d e^{m_{i+1}} \dots + \dots d e^{m_{i-1}+m_i} d e^{m_{i+1}} \dots) \theta(m_i) \\ &= \sum_{i=1}^L \text{tr} (\dots d e^{m_i-1} d e^{m_i} d e^{m_{i+1}} \dots + \dots d e^{m_{i-1}} d e^{m_i+m_{i+1}} \dots) \theta(m_i) \end{aligned} \quad (7.46)$$

which is obviously an identity.

However we note that the condition (7.43) is not necessary. It would be enough to demand the weaker condition $u(m)de^md = de^{m-1}d$, respectively $f(m)^{-1}de^md = d^2$. This simpler algebra has the representation

$$d = \begin{pmatrix} 1 & f(1) & f(2) & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \quad (7.47)$$

defined through the single-site weights (7.39) of the corresponding ZRP. We briefly proof this first result: The matrix product becomes: $F(m_1, m_2, \dots, m_N) = \text{tr} \prod_{\nu=1}^N de^{m_\nu}$. On executing the terms de^{m_ν} under the product one eventually finds that $F(\{m_\nu\}) = \text{tr} \prod_{\nu=1}^N \sum_{i \geq 1} |1\rangle \langle i| f(m_\nu + i - 1)$. The trace over such a product clearly gives (7.38).

So the one-dimensional solution (7.38) (simply given by scalar one-site factors (7.39)) of the ZRP with infinite state space can be transformed to an infinite-dimensional solution of the ASEP with two-dimensional state space.

7.3.2 A generalization to open boundaries

This can now be used for more involved generalizations of the process. For example consider the following open-boundary version of the model:

$$1 \overbrace{0 \dots 0}^m \rightarrow 01 \overbrace{0 \dots 0}^{m-1}, \quad \text{with rate } u(m), \quad (7.48)$$

$$|0 \rightarrow |1, \quad \text{with rate } \alpha, \quad (7.49)$$

$$|1 \rightarrow |0, \quad \text{with rate } \beta. \quad (7.50)$$

For this model we can write

$$F(\tau_1, \tau_2, \dots, \tau_L) = \langle \tilde{W} | \prod_{l=1}^L [\tau_l \tilde{D} + (1 - \tau_l) \tilde{E}] | \tilde{V} \rangle. \quad (7.51)$$

This equation indeed gives the steady state if the matrices fulfill the (infinite-dimensional) algebra

$$u(m) \tilde{D} \tilde{E}^m (\tilde{D}, |\tilde{V}\rangle) = (\tilde{D} + \tilde{E}) \tilde{E}^{m-1} (\tilde{D}, |\tilde{V}\rangle), \quad (7.52)$$

$$\alpha \langle \tilde{W} | \tilde{E} = \langle \tilde{W} |, \quad (7.53)$$

$$\beta \tilde{D} |\tilde{V}\rangle = |\tilde{V}\rangle, \quad (7.54)$$

which generalizes the DEHP-algebra. The expression $(\tilde{D}, |\tilde{V}\rangle)$ is short-hand for either \tilde{D} or $|\tilde{V}\rangle$. We note that a different generalization to open boundaries leading to a factorizing steady state has been investigated in [97].

That (7.52-7.54) does solve the process can be seen by considering the master equation. Take for instance configurations of the form $|1^{m_1}, 0^{n_1}, 1^{m_2}, 0^{n_2}\rangle$ as in [59]:

$$\begin{aligned} [u(n_1) + u(n_2)] \langle \tilde{W} | \tilde{D}^{m_1} \tilde{E}^{n_1} \tilde{D}^{m_2} \tilde{E}^{n_2} | \tilde{V} \rangle &= \alpha \langle \tilde{W} | \tilde{D}^{m_1-1} \tilde{E}^{n_1} \tilde{D}^{m_2} \tilde{E}^{n_2} | \tilde{V} \rangle \\ &+ u(1) \langle \tilde{W} | \tilde{D}^{m_1} \tilde{E}^{n_1-1} \tilde{D} \tilde{E} \tilde{D}^{m_2-1} \tilde{E}^{n_2} | \tilde{V} \rangle \\ &+ \beta \langle \tilde{W} | \tilde{D}^{m_1} \tilde{E}^{n_1} \tilde{D}^{m_2} \tilde{E}^{n_2-1} \tilde{D} | \tilde{V} \rangle. \end{aligned} \quad (7.55)$$

Inserting for each term with an α , β or an $u(n)$ the corresponding expression easily gives that the equation is fulfilled. This can be extended to all other possible configurations. The full proof gives no new insights and is just omitted here.

Normally it would be more natural to have $\alpha \rightarrow \alpha(m)$. However this process is not solvable in the same ‘simple’ way as the case of constant α . But one might think instead of a street for example of fixed length L and cars entering perpendicular to the street at the one end so that drivers can only see the first ‘site’ and enter with probability α . Once entered they now see the next car $m + 1$ lattice spacings in front and drive always with rate $u(m)$. At the end of the street they leave perpendicular to the road with rate β into the next road. So this for example models the movement of cars in a Manhattan geometry of streets.

Or it may simulate a part of a freeway with on and off-ramp where cars can not see the situation on the freeway before entering so that they enter with constant rate.

7.3.3 Competition between zero-range and normal exclusion on the ring

The same structure of solution has the corresponding two-species process on a ring:

$$\overbrace{10\dots 0}^m \rightarrow 01\overbrace{0\dots 0}^{m-1}, \quad \text{with rate } u(m), \quad (7.56)$$

$$20 \rightarrow 02, \quad \text{with rate } \alpha, \quad (7.57)$$

$$12 \rightarrow 21, \quad \text{with rate } \beta, \quad (7.58)$$

by taking $|V\rangle\langle W| =: A$ and introducing the matrix representing a 2-particle. Here we have 1-particles whose movement depends precisely on the free space in front and 2-particles that only look if the next site in front is free. In addition 1-particles are allowed to overtake 2-particles with a constant rate. This dynamics is a sort of competition between zero-range dynamics of 1-particles and normal asymmetric exclusion dynamics of two-particles. Since both the ZRP and the ASEP can lead to phase transitions it would be interesting to study the influence of this competition on the phase diagram.

7.3.4 A mass-transport model on the ring

The two-species Model can now be mapped onto a mass-transport model by mapping

$$0 \rightarrow 00, \quad (7.59)$$

$$2 \rightarrow 01 \quad (7.60)$$

and afterwards mapping

$$\text{holes} \rightarrow \text{particles}, \quad (7.61)$$

$$\text{particles} \rightarrow \text{sites}. \quad (7.62)$$

Then the dynamics in the mass-transport model is defined by the rates

$$u_l(m) = \begin{cases} u(m), & \text{for } l = 2 \text{ and } m = 2, 4, 6, \dots, \\ \alpha, & \text{for } l = 2 \text{ and } m = 3, 5, \dots, \\ \beta, & \text{for } l = 1 \text{ and } m = 1. \end{cases} \quad (7.63)$$

This is a generalization of the excess-mass process defined by rates (3.1-3.2 where $u_2(m) = \alpha = 1$). This is the most general generalization of the excess-mass process that we found. The more natural case $\alpha = \alpha(m)$ also has some structure but could up to now not be solved in terms of a sort of matrix product.

7.4 A three-state model

The effect of excess-mass formation occurs also in other models. We consider a driven-diffusive system on a one-dimensional lattice with L sites, labeled $i = 1, 2, \dots, L$ with periodic boundary conditions. Each site can now be in three different states: either a site i is empty ($n_i = 0$) or occupied by one ($n_i = 1$) or two particles ($n_i = 2$). The hopping is completely asymmetric and the process evolves in continuous time. Thus the transitions $(s_i s_{i+1}) \longrightarrow (s'_i s'_{i+1})$ on sites i and $i + 1$ are

$$10 \longrightarrow 01, \quad \text{with rate } u(1, 0), \quad (7.64)$$

$$11 \longrightarrow 02, \quad \text{with rate } u(1, 1), \quad (7.65)$$

$$20 \longrightarrow 11, \quad \text{with rate } u(2, 0), \quad (7.66)$$

$$\longrightarrow 02, \quad \text{with rate } \tilde{u}(2, 0), \quad (7.67)$$

$$21 \longrightarrow 12, \quad \text{with rate } u(2, 1). \quad (7.68)$$

The Hamilton operator H is a sum of L local hamiltonians, acting on two neighboring sites:

$$H = \sum_{i=1}^L h_{i,i+1}. \quad (7.69)$$

The local hamiltonian in the standard basis is given as

$$h = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u(1, 0) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u(1, 1) & 0 & -\tilde{u}(2, 0) & 0 & 0 & 0 \\ 0 & 0 & 0 & -u(1, 0) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -u(1, 1) & 0 & u(2, 0) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & u(2, 1) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -u(2, 0) - \tilde{u}(2, 0) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -u(2, 1) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (7.70)$$

For $|P_L\rangle$ to be stationary, the master equation has to be fulfilled which can be rewritten in form of a Schrödinger equation:

$$H|P_L\rangle = 0. \quad (7.71)$$

To solve this equation we perform a matrix- (or more generally tensor-) product ansatz. This implies that the steady state probability of any configuration of a system with L cells can be written as

$$|P_L\rangle = Z^{-1} \text{Tr} \left(\begin{pmatrix} \mathcal{E} \\ \mathcal{A} \\ \mathcal{D} \end{pmatrix}^{\otimes L} \right), \quad (7.72)$$

where the operator \mathcal{E} represents an empty cell and the operators \mathcal{A} , \mathcal{D} a cell occupied by one, two particles, respectively.

To fulfill this we introduce the following canceling mechanism

$$h \begin{pmatrix} \mathcal{E} \\ \mathcal{A} \\ \mathcal{D} \end{pmatrix} = \begin{pmatrix} \hat{\mathcal{E}} \\ \hat{\mathcal{A}} \\ \hat{\mathcal{D}} \end{pmatrix} \otimes \begin{pmatrix} \mathcal{E} \\ \mathcal{A} \\ \mathcal{D} \end{pmatrix} - \begin{pmatrix} \mathcal{E} \\ \mathcal{A} \\ \mathcal{D} \end{pmatrix} \otimes \begin{pmatrix} \hat{\mathcal{E}} \\ \hat{\mathcal{A}} \\ \hat{\mathcal{D}} \end{pmatrix}. \quad (7.73)$$

This leads to the following algebra

$$\bar{\mathcal{E}}\mathcal{E} - \mathcal{E}\bar{\mathcal{E}} = 0, \quad (7.74)$$

$$\bar{\mathcal{E}}\mathcal{A} - \mathcal{E}\bar{\mathcal{A}} = u(1, 0)\mathcal{A}\mathcal{E}, \quad (7.75)$$

$$\bar{\mathcal{E}}\mathcal{D} - \mathcal{E}\bar{\mathcal{D}} = u(1, 1)\mathcal{A}\mathcal{A} + \tilde{u}(2, 0)\mathcal{D}\mathcal{E}, \quad (7.76)$$

$$\bar{\mathcal{A}}\mathcal{E} - \mathcal{A}\bar{\mathcal{E}} = -u(1, 0)\mathcal{A}\mathcal{E}, \quad (7.77)$$

$$\bar{\mathcal{A}}\mathcal{A} - \mathcal{A}\bar{\mathcal{A}} = -u(1, 1)\mathcal{A}\mathcal{A} + u(2, 0)\mathcal{D}\mathcal{E}, \quad (7.78)$$

$$\bar{\mathcal{A}}\mathcal{D} - \mathcal{A}\bar{\mathcal{D}} = u(2, 1)\mathcal{D}\mathcal{A}, \quad (7.79)$$

$$\bar{\mathcal{D}}\mathcal{E} - \mathcal{D}\bar{\mathcal{E}} = -(u(2, 0) + \tilde{u}(2, 0))\mathcal{D}\mathcal{E}, \quad (7.80)$$

$$\bar{\mathcal{D}}\mathcal{A} - \mathcal{D}\bar{\mathcal{A}} = -u(2, 1)\mathcal{D}\mathcal{A}, \quad (7.81)$$

$$\bar{\mathcal{D}}\mathcal{D} - \mathcal{D}\bar{\mathcal{D}} = 0. \quad (7.82)$$

Then it is possible to calculate for example the flow:

$$J_L = [\text{Tr } \mathcal{C}^L]^{-1} u(1, 0) \text{Tr } (\mathcal{A}\mathcal{E}\mathcal{C}^{L-2}) + u(1, 1) \text{Tr } (\mathcal{A}\mathcal{A}\mathcal{C}^{L-2}) \quad (7.83)$$

$$+ [2\tilde{u}(2, 0) + u(2, 0)] \text{Tr } (\mathcal{D}\mathcal{E}\mathcal{C}^{L-2}) + u(2, 1) \text{Tr } (\mathcal{D}\mathcal{A}\mathcal{C}^{L-2}) \quad (7.84)$$

with $\mathcal{C} = \mathcal{E} + \mathcal{A} + \mathcal{D}$.

It is easy to check that the model has a factorized steady state, i.e. a one-dimensional representation of the algebra iff

$$u(2, 0) + \tilde{u}(2, 0) = u(1, 0) + u(2, 1). \quad (7.85)$$

Thus $u(1, 1)$ is a free parameter. In the cases in which (7.85) is fulfilled, we have

$$\bar{\mathcal{A}} = \mathcal{A} \left[\frac{\bar{\mathcal{E}}}{\mathcal{E}} - u(1, 0) \right], \quad (7.86)$$

$$\bar{\mathcal{D}} = \mathcal{D} \left[\frac{\bar{\mathcal{E}}}{\mathcal{E}} - u(2, 0) - \tilde{u}(2, 0) \right]. \quad (7.87)$$

The case $u(2, 0) = 0$

This is the case where two particles once occupying the same site will stay together for all times, since one of them can not move alone. Let all other rates be larger than zero. Since $u(1, 1) > 0$ two-particle occupations are created while single-particle occupations can not be created due to $u(2, 0) = 0$. Obviously this leads for an odd number of particles on the lattice to an excess-mass formation in the form of a single particle at one site.

Let us have a closer look at the solution: With $\hat{\mathcal{E}} = \mathcal{E} + 1$, $\hat{\mathcal{D}} = \mathcal{D} - 1$ and $\hat{\mathcal{A}} = \mathcal{A}$ we obtain the following algebra:

$$\mathcal{D}\mathcal{E} = \mathcal{D} + \mathcal{E}, \quad (7.88)$$

$$u(1, 0)\mathcal{A}\mathcal{E} = \mathcal{A}, \quad (7.89)$$

$$u(2, 1)\mathcal{D}\mathcal{A} = \mathcal{A}, \quad (7.90)$$

$$u(1, 1)\mathcal{A}\mathcal{A} = 0. \quad (7.91)$$

The term $\mathcal{A}\mathcal{A} = 0$ makes sure that not more than a single site can support a single particle. This algebra has the same form as that of the process with excess-mass formation (sec. 3). The main difference is that here it makes sense to distinguish between the rates $\tilde{u}(2, 0)$ and $u(1, 0)$, while in the ASEP with excess-mass formation there is no reason why a particle

with an odd-length gap to the left should move at a different rate to the right as a particle that finds an even-length to its left. The steady state obviously is the same as for the defect ASEP (sec. 2.2.3) and one can take over the phase diagram with all four phases. The equivalence is simply: This process can now be mapped to our model by interpreting

- occupation 0 corresponds to a hole,
- occupation 1 corresponds to a the single defect,
- occupation 2 corresponds to a usual particle.

Note that recently some progress has been made on three-state proceses. Another generalization of the DEHP algebra has been found [98].

The case $u(2, 0) = u(1, 1) = 0$

The difference to the case considered above is that here also $u(1, 1) = 0$. As a consequence, the number of sites occupied by 0, 1 or 2 particles is conserved in time. Thus if we have an initial configuration that contains only 0s and 1s or 2s and 0s or 2s and 1s the system evolves simply under usual ASEP dynamics. From the ASEP on a ring we know that the stationary probability measure is uniform (2.14), i.e. all configurations (with constant particle number) are equally likely. However the model becomes more interesting if one starts with a configuration that contains 0s, 1s and 2s as well:

Taking in (7.74-7.82) $u(2, 0) = u(1, 1) = 0$ and $\hat{\mathcal{E}} = 1, \hat{\mathcal{A}} = 0, \hat{\mathcal{D}} = -1$ and setting $u_2(2, 0) = 1$ which gives just a rescaling of time, one obtains (7.88-7.90). Since the number of all occupations are fixed, this is equivalent to an ASEP with two species of particles:

- occupation 0 corresponds to a hole,
- occupation 1 corresponds to a particle of the second species,
- occupation 2 corresponds to a particle of the first species.

The number of 1-species particles is in the steady state therefore $\lfloor N/2 \rfloor$ and the number of 2-species particles is $N \bmod 2$. We note that every configuration can be reached and we thus have ergodic dynamics and a unique stationary probability measure.

This is mentioned because in the three-state process one simple change of a parameter leads leads to the fact that the system can support an arbitrary number of defects. In contrast, the ASEP with excess-mass formation always reached a steady state with a single defect in the form of a single odd-length gap.

The case $u(1, 1) = 0$

When the only vanishing rate is $u(1, 1)$ obviously two-particle occupations can not be created instead of one-particle occupations. The system tries to reach a homogeneous state with one-particle occupations. If $N > L$ there is no empty site and if $N < L$ there is no site occupied by two particles. Setting $\hat{E} = E$, $\hat{D} = -u(2, 1)A$, and $\hat{A} = (1 - u(1, 0))E$, we find

$$A = \begin{pmatrix} u(2, 1)^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (7.92)$$

We take now $u(1, 0) = 1$ without loss of generality. Then the matrices fulfill the algebra

$$AE = EA = E, \quad (7.93)$$

$$DE = ED = 0, \quad (7.94)$$

$$DA = AD = u(2, 1)^{-1}D. \quad (7.95)$$

This algebra can easily be understood. We know that a configuration that contains a 2 and a 0 has weight zero and all other configurations have equal weight. So The first and third rule transport an E and D respectively through the chain (and delete a A since it is unimportant for the weight). If a D ‘meets’ an E the second rule thus gives weight 0.

Finally we just mention that the model can also be considered as a two-component creation/annihilation process, see [99] and references therein. Then the exact solutions of the previous sections correspond to the cases i) no creation and annihilation, ii) no creation, and iii) no annihilation.

8 Summary

This work mainly deals with a generalization of the (totally) asymmetric simple exclusion process (ASEP) with periodic boundary conditions [4]. The ASEP is defined on a one-dimensional discrete lattice in which every site can be occupied by exactly one particle or it may be empty. In the original definition particles can move one site to the right if the target site is empty. Here we consider the possibility of moving either one or two sites. Both random-sequential and parallel dynamics are investigated. For a special limit of parameters, where particles always move as fast as possible with respect to their maximum velocity and availability of empty space ahead, the process evolves into special regions of the configuration space. This is what we refer to as ASEP with excess-mass formation: The number of empty sites between particles is always a multiple of the maximum velocity two. However depending on particle number and lattice size there can remain a single excess hole. The quantity of interest is the stationary probability for a certain configuration in the frame of this excess hole. One observes two phases where the velocity takes different values and serves as an order parameter for the first-order transition. It turns out that the excess hole plays the role of a defect. For continuous time the process recovers the ASEP with a single defect particle [23] and for parallel update it yields a natural new defect dynamics. In phase 1 the typical density profiles have exponentially damped oscillations around the defect and a constant density in the bulk. In phase two the profile decreases in front of the defect algebraically and is accordingly lowered behind. This is a typical form of a limiting shock profile with equal densities to the left and right.

The mathematical framework that is used to obtain exact results for this process and related models is the the matrix-product ansatz [15]. As a by-product we show that the stationary weights of the ASEP with parallel update and open boundaries can be written as a product of a scalar pair-factorized and a matrix-product state. This replaces the known matrix-in-matrix form [24] and suggests that there is a general relationship between matrix-product states for random-sequential dynamics, simple discrete-time updates as ordered sequential and sublattice-parallel dynamics [13] and the parallel update which includes an additional nearest-neighbor correlation [26].

For the ASEP with maximum velocity two and a more general definition we obtain the exact form of the steady state for two particles on a ring of arbitrary size. This result already suggests that there can in general be oscillations in the distribution of headway (which is understood as the empty space available in front of a particle). A mean-field theory is developed that takes certain long-ranged correlations into account. One observes a remarkably good agreement with computer simulations. For the ASEP with excess-mass formation the theory predicts the exact steady state for the thermodynamic limit since the single defect does not affect the overall distribution of headway. This observation leads to a statement of exactness of headway distribution for a broader class of models where the configurations have no product measure.

Finally we study related models: A two-species zero-range process and a three-state model with matrix-product state related to the ASEP with excess-mass formation and a generalization for further studies are introduced.

A Alternative form of the master equation

This appendix refers to the ASEP with excess-mass formation and parallel dynamics. The master equation was written in with the indexed operators (5.54). We rewrite the master equation it in terms of E , D and A , once because (5.53) might be a little bit unfamiliar and once to show the advantage in simplicity of this equation.

In the present model as well as in the ASEP with parallel update and open boundaries the complexity is reduced if one rewrites the weight of a configuration with indexed operators $D_1, E_1, \langle W_1 |, | V_1 \rangle$. The correlations due to the parallelism included in the formulation with un-indexed operators are no longer present in the algebra for the indexed operators which is a main importance. If we now try to write the master equation in terms of un-indexed operators it is natural that these correlations come into play. Consider the bulk product on the right-hand side in section 5.2.

$$\prod_{\mu=1}^{N-1} \left(E_1^{n_\mu} D_1 + \frac{p}{1-p} \theta(n_\mu) E_1^{n_\mu-1} D_1 E_1 \right) \quad (\text{A.1})$$

We have the relations $(1 - p\theta(n))DE^nD = D_1E_1^nD_1$ and $(1 - p\theta(n))AE^nD = A_1E_1^nD_1$. Consider for example terms with $E_1^{n_\mu-1}D_1E_1$ in the product. On transforming this one has not generally a factor $(1 - p(\theta(n_\mu - 1)))$ since there are terms in the product in which it couples to $E_1^{n_{\mu-1}-1}D_1E_1$ to the left, leading to a matrix product $E_1^{n_\mu-1}D_1E_1^{n_\mu}D_1E_1$ which has to be weighted by $(1 - p\theta(n_\mu))$. Generally the product can be written as

$$\prod_{\mu=1}^{N-1} \left(E_1^{n_\mu} D_1 + \frac{p}{1-p} \theta(n_\mu) E_1^{n_\mu-1} D_1 E_1 \right) = \sum_{\sigma_1=0}^1 \cdots \sum_{\sigma_N=0}^1 \left[(1 - \sigma_1) \prod_{\mu=1}^{N-1} (1 - p\theta(n_\mu + \sigma_\mu))^{1-\sigma_{\mu+1}} \left(p\theta(n_\mu) \frac{1 - p\theta(n_\mu - 1 + \sigma_\mu)}{1-p} \right)^{\sigma_{\mu+1}} E^{n_\mu - \sigma_{\mu+1} + \sigma_\mu} D \right] E^{\sigma_N} \quad (\text{A.2})$$

Here the correlations in the hopping of neighboring particles has to be taken into account through variables σ_μ . Concrete:

$$\sigma_\mu = \begin{cases} 1, & \text{if particle } \mu \text{ moves,} \\ 0, & \text{otherwise.} \end{cases} \quad (\text{A.3})$$

In the master equation of the form (5.53) to the left of the product occur either factors A (here the particle does not move, so $\sigma_1 = 0$) or AE (here the first particle moves, so $\sigma_1 = 1$). This leads after simplifications to

$$\begin{aligned} E_1^{\sigma_1} \prod_{\mu=1}^{N-1} \left(E_1^{n_\mu} D_1 + \frac{p}{1-p} \theta(n_\mu) E_1^{n_\mu-1} D_1 E_1 \right) &= \\ = \sum_{\sigma_\mu \in \{0,1\}_{\mu=1..N}} \prod_{\mu=1}^{N-1} \left[p^{\sigma_{\mu+1}} \theta(n_\mu)^{\sigma_{\mu+1}} (1 - p\theta(n_\mu - \sigma_{\mu+1}))^{1-\sigma_\mu} E^{n_\mu - \sigma_{\mu+1} + \sigma_\mu} D \right] E^{\sigma_N} (1 - p\sigma_N)^{-1}. \end{aligned} \quad (\text{A.4})$$

Using this the master equation is rewritten in terms of E, D, A as

$$\begin{aligned}
& \text{tr} A \prod_{\mu=1}^{N-1} [E^{n_\mu} D] E^{n_N} = \sum_{\sigma_\mu, \mu=1..N} \text{tr} A [\\
& (1 - \sigma_1) \prod_{\mu=1}^{N-1} \left[p^{\sigma_{\mu+1}} \theta(n_\mu)^{\sigma_{\mu+1}} (1 - p\theta(n_\mu - \sigma_{\mu+1}))^{1-\sigma_\mu} E^{n_\mu - \sigma_{\mu+1} + \sigma_\mu} D \right] E^{n_N + \sigma_N} \\
& \times (1 - \beta\delta_{n_N,0} - p\theta(n_N))^{1-\sigma_N} \\
& + \delta_{n_1,0}\beta(1 - \sigma_1)(1 - \sigma_2) \prod_{\mu=2}^{N-1} \left[p^{\sigma_{\mu+1}} \theta(n_\mu)^{\sigma_{\mu+1}} (1 - p\theta(n_\mu - \sigma_{\mu+1}))^{1-\sigma_\mu} E^{n_\mu - \sigma_{\mu+1} + \sigma_\mu} D \right] E^{n_N + \sigma_N} D \\
& \times (1 - p\theta(n_N))^{1-\sigma_N} \\
& + \delta_{n_1,1}p\beta\sigma_1\sigma_2 \prod_{\mu=2}^{N-1} \left[p^{\sigma_{\mu+1}} \theta(n_\mu)^{\sigma_{\mu+1}} (1 - p\theta(n_\mu - \sigma_{\mu+1}))^{1-\sigma_\mu} E^{n_\mu - \sigma_{\mu+1} + \sigma_\mu} D \right] E^{n_N + \sigma_N} D \\
& \times (1 - p\theta(n_N))^{1-\sigma_N} \\
& + p\sigma_1 \prod_{\mu=1}^{N-1} \left[p^{\sigma_{\mu+1}} \theta(n_\mu)^{\sigma_{\mu+1}} (1 - p\theta(n_\mu - \sigma_{\mu+1}))^{1-\sigma_\mu} E^{n_\mu - \sigma_{\mu+1} + \sigma_\mu} D \right] E^{n_N + \sigma_N - 1} \\
& \times (1 - p)^{1-\sigma_N} \theta(n_N) \Big]. \tag{A.5}
\end{aligned}$$

This equation is quite easily understood. Consider the bulk at first: For particle μ to have $2n_\mu$ holes in front it must have had $2(n_\mu - \sigma_{\mu+1} + \sigma_\mu)$ holes in front before the update. The factor $\theta(n_\mu)^{\sigma_{\mu+1}}$ gives zero only if $n_\mu = 0$ and at the same time $\sigma_{\mu+1} = 1$ excluding impossible cases. The other factors are the correct transition probabilities. Of course this relation can also be obtained directly from working out the transfer matrices in terms of un-indexed matrices. The factors for the boundaries can be explained in a similar fashion. Check equation (A.5) for $N = 2$:

$$\begin{aligned}
\text{tr} A E^{n_1} D E^{n_2} &= (1 - p\theta(n_1)) (1 - \beta\delta_{n_2,0} - p\theta(n_2)) \text{tr} A E^{n_1} D E^{n_2} \\
&+ p\theta(n_1) (1 - p\theta(n_1 - 1)) \text{tr} A E^{n_1-1} D E^{n_2+1} \\
&+ \delta_{n_1,0}\beta (1 - p\theta(n_2)) \text{tr} A E^{n_2} D \\
&+ \delta_{n_1,1}p\beta \text{tr} A E^{n_2+1} D \\
&+ p(1 - p)\theta(n_2) \text{tr} A E^{n_1+1} D E^{n_2-1} \\
&+ p^2\theta(n_1)\theta(n_2) \text{tr} A E^{n_1} D E^{n_2}. \tag{A.6}
\end{aligned}$$

One can check that on transforming this equation into an equation with E_1, D_1, A_1 one obtains the same equation as resulting from (5.53). So one should be familiar with the fact that the two forms of the master equation (5.53) and (A.5) are completely equivalent.

B COMF equations for the parallel ASEP with excess-mass formation

The COMF equations for the probability $P(n)$ to find n empty sites in front of a particle read here

$$P(0) = g_0 P(0) + \beta_1 g_0 P(1) + p_2 g_0 P(2), \quad (\text{B.1})$$

$$\begin{aligned} P(1) &= g_1 P(0) + (\beta_0 g_0 + \beta_1 g_1) P(1) \\ &+ (p_1 g_0 + p_2 g_1) P(2) + p_2 g_0 P(3), \end{aligned} \quad (\text{B.2})$$

$$\begin{aligned} P(2) &= g_2 P(0) + (\beta_0 g_1 + \beta_1 g_2) P(1) \\ &+ (p_0 g_0 + p_1 g_1 + p_2 g_2) P(2) \\ &+ (p_1 g_0 + p_2 g_1) P(3) + p_2 g_0 P(4), \end{aligned} \quad (\text{B.3})$$

$$\begin{aligned} P(3) &= \beta_0 g_2 P(1) + (p_0 g_1 + p_1 g_2) P(2) \\ &+ (p_0 g_0 + p_1 g_1 + p_2 g_2) P(3) \\ &+ (p_1 g_0 + p_2 g_1) P(4) + p_2 g_0 P(5), \end{aligned} \quad (\text{B.4})$$

$$\begin{aligned} P(m) &= p_0 g_2 P(m-2) + (p_0 g_1 + p_1 g_2) P(m-1) \\ &+ (p_0 g_0 + p_1 g_1 + p_2 g_2) P(m) \\ &+ (p_1 g_0 + p_2 g_1) P(m+1) + p_2 g_0 P(m+2), \text{ for } m \geq 4. \end{aligned} \quad (\text{B.5})$$

with

$$g_0 = P(0) + \beta_0 P(1) + p_0 s, \quad (\text{B.6})$$

$$g_1 = \beta_1 P(1) + p_1 s, \quad (\text{B.7})$$

$$g_2 = p_2 s, \quad (\text{B.8})$$

$$s = 1 - P(0) - P(1). \quad (\text{B.9})$$

Additionally one has $g_0 + g_1 + g_2 = 1$.

Bibliography

- [1] V. Privman, editor. *Nonequilibrium Statistical Mechanics in One Dimension*. Cambridge University Press, 1997.
- [2] D. Chowdhury, L. Santen, and A. Schadschneider. Statistical physics of vehicular traffic and some related systems. *Physics Reports*, 329:199–329, 2000.
- [3] A. Schadschneider D. Chowdhury and K. Nishinari. Physics of transport and traffic phenomena in biology: from molecular motors and cells to organisms. *Phys. Life Revs.*, 2:213, 2005.
- [4] B. Derrida. An exactly soluble non-equilibrium system: The asymmetric exclusion process. *Phys. Rep.*, 301:65, 1998.
- [5] C. Appert-Roland, F. Chevoir, P. Gondret, S. Lassarre, J.P. Lebacque, and M. Schreckenberg, editors. *Traffic and Granular Flow 07*. Springer, 2009.
- [6] A. Schadschneider, T. Pöschel, R. Kühne, M. Schreckenberg, and D. E. Wolf, editors. *Traffic and Granular Flow 05*. Springer, 2007.
- [7] S. Wolfram. *A New Kind of Science*. Wolfram Media, Inc, Champaign, Illinois, USA, 2002.
- [8] K. Nagel and M. Schreckenberg. A cellular automaton model for freeway traffic. *J. Physique I*, 2:2221–2229, 1992.
- [9] M. Woelki, A. Schadschneider and M. Schreckenberg. Asymmetric exclusion processes with shuffled dynamics. *J. Phys. A*, 39:33, 2006.
- [10] M. Woelki. Steady states of discrete mass transport models. Master’s thesis, University Duisburg-Essen, 2005.
- [11] M. Woelki, Andreas Schadschneider, and M. Schreckenberg. Fundamental diagram of a one-dimensional cellular automaton model for pedestrian flow. In Waldau et al. [12], pages 423–428.
- [12] N. Waldau, P. Gattermann, H. Knoflacher, and M. Schreckenberg, editors. *Pedestrian and Evacuation Dynamics 05*. Springer, 2007.
- [13] N. Rajewsky, L. Santen, A. Schadschneider, and M. Schreckenberg. The asymmetric exclusion process: Comparison of update procedures. *J. Stat. Phys.*, 92:151, 1998.
- [14] L. Santen. Numerical investigations of discrete models for traffic flow. *Dissertation, Universität zu Köln, Germany*, 1999.
- [15] R. A. Blythe and M. R. Evans. Nonequilibrium steady states of matrix product form: A solver’s guide. *J. Phys. A*, 40:333, 2007.

- [16] M. R. Evans. Phase transitions in one-dimensional non-equilibrium systems. *Braz. J. Phys.*, 30:42, 2000.
- [17] H. Spohn. *Large Scale dynamics of Interacting Particles*. Springer, Berlin, 1991.
- [18] O. Golinelli and K. Mallick. The asymmetric simple exclusion process: an integrable model for non-equilibrium statistical mechanics. *J. Phys. A*, 39:12679, 2006.
- [19] M. R. Evans and T. Hanney. Non-equilibrium statistical mechanics of the zero-range process and related models. *J. Phys. A*, 38:R195, 2005.
- [20] B. Derrida, E. Domany, and D. Mukamel. An exact solution of an one-dimensional asymmetric exclusion model with open boundaries. *J. Phys. A*, 26:667, 1992.
- [21] B. Derrida, M. R. Evans, V. Hakim, and V. Pasquier. Exact solution of a 1d asymmetric exclusion model using a matrix formulation. *J. Phys. A*, 26:1493, 1993.
- [22] B. Derrida, S. A. Janowsky, J. L. Lebowitz, and E. R. Speer. Exact solution of the totally asymmetric exclusion process: Shock profiles. *J. Stat. Phys.*, 73:813, 1993.
- [23] K. Mallick. Shocks in the asymmetry exclusion model with an impurity. *J. Phys. A*, 29:5375, 1996.
- [24] M. R. Evans, N. Rajewsky, and E. R. Speer. Exact solution of a cellular automaton for traffic. *J. Stat. Phys.*, 95:45–96, 1999.
- [25] J. de Gier and B. Nienhuis. Exact stationary state for an asymmetric exclusion process with fully parallel dynamics. *Phys. Rev. E*, 59:4899–4911, 1999.
- [26] M. Schreckenberg, A. Schadschneider, K. Nagel, and N. Ito. Discrete stochastic models for traffic flow. *Phys. Rev. E*, 51:2939–2949, 1995.
- [27] N. Rajewsky, A. Schadschneider, and M. Schreckenberg. The asymmetric exclusion model with sequential update. *J. Phys. A*, 29(12):L305, 1996.
- [28] A. Honecker and I. Peschel. Matrix-product states for a one-dimensional lattice gas with parallel dynamics. *J. Stat. Phys.*, 88:319–345, 1997.
- [29] H. Hinrichsen. Matrix-product ground states for exclusion processes with parallel dynamics. *J. Phys. A*, 29:3659–3667, 1996.
- [30] M. Woelki and M. Schreckenberg. Exact matrix-product states for parallel dynamics: open boundaries and excess-mass on the ring. *J. Stat. Mech.*, page P05014, 2009.
- [31] M. Woelki and M. Schreckenberg. Headway oscillations and phase transitions for diffusing particles with increased velocity. *J. Phys. A (in press)*, 2009.
- [32] M. Woelki and M. Schreckenberg. Phase transitions and even/odd effects in asymmetric exclusion models. In Appert-Roland et al. [5], pages 435–440.
- [33] T. M. Liggett. *Interacting Particle Systems*. Springer, New York, 1985.
- [34] M. Woelki, A. Schadschneider, and M. Schreckenberg. Asymmetric exclusion processes with non-factorizing steady states. In Schadschneider et al. [6], pages 473–479.

- [35] M. Evans. Phase transitions in stochastic models of flow. In Schadschneider et al. [6], pages 447–459.
- [36] B. Waclaw, J. Sopik, W. Janke, and H. Meyer-Ortmanns. Pair-factorized steady states on a ring. *eprint arXiv: 0901.3664*, 2009.
- [37] B. Derrida, S. A. Janowsky, J. L. Lebowitz, and E. R. Speer. Microscopic shock profiles: Exact solution of a non-equilibrium system. *Europhys. Lett.*, 22:651–656, 1993.
- [38] B. Derrida, J. L. Lebowitz, and E. R. Speer. Shock profiles for the asymmetric exclusion process in one dimension. *J. Stat. Phys.*, 89:135–167, 1997.
- [39] P. A. Ferrari, L. R. G. Fontes, and Y. Kohayakawa. Invariant measures for a two species asymmetric process. *J. Stat. Phys.*, 76:1153, 1994.
- [40] G. M. Schuetz. Critical phenomena and universal dynamics in one-dimensional driven diffusive systems with two species of particles. *J. Phys. A*, 36:R339–R379, 2003.
- [41] F. Alexander, S.A. Janowsky, J.L. Lebowitz, and H.A. van Beijeren. Shock fluctuations in one-dimensional lattice fluids. *Phys. Rev. E*, 47:403–410, 1993.
- [42] M. R. Evans. Bose-Einstein condensation in disordered exclusion models and relation to traffic flow. *Europhys. Lett.*, 36:13–18, 1996.
- [43] H.W. Lee, V. Popkov, and D. Kim. Two-way traffic flow: Exactly solvable model of traffic jam. *J. Phys. A*, 30:8497, 1997.
- [44] M.E. Fouladvand and H.-W. Lee. Exactly solvable two-way traffic model with ordered sequential update. *Phys. Rev. E*, 60(6):6465–6479, 1999.
- [45] Vladislav Popkov and Ingo Peschel. Exactly solvable statistical model for two-way traffic. *J. Phys. A*, 33:3989–3995, 2000.
- [46] F. R. Gantmacher. *Matrix Theory*. Chelsea, New York, 1964.
- [47] v. Hakim and J.-P. Nadal. Exact results for 2d directed animals on a strip of finite width. *J. Phys. A*, 16:213–218, 1983.
- [48] A. Klümper, A. Schadschneider, and J. Zittartz. Equivalence and solution of anisotropic spin-1 models and generalized t-J fermion models in one dimension. *J. Phys. A*, 24:955, 1991.
- [49] H. Hinrichsen, S. Sandow, and I. Peschel. On matrix-product ground states for reaction diffusion models. *J. Phys. A*, 29:2643, 1996.
- [50] K. Klauck. *On low-dimensional stochastic models*. PhD thesis, Universität zu Köln, 2000.
- [51] K. Mallick, S. Mallick, and N Rajewsky. Exact solution of an exclusion process with three classes of particles and vacancies. *J. Phys. A*, 32:8399, 1999.
- [52] F. Spitzer. Interaction of Markov processes. *Advances in Math.*, 5:246–290, 1970.
- [53] C. Domb and J. L. Lebowitz, editors. *Statistical Mechanics of Driven Diffusive Systems*, volume 17. Academic Press, London, 2000.

- [54] S. Katz, J. L. Lebowitz, and H. Spohn. Phase transitions in stationary non equilibrium states of model lattice systems. *Phys. Rev. B*, 28:1655–1658, 1983.
- [55] S. Katz, J. L. Lebowitz, and H. Spohn. Stationary nonequilibrium states for stochastic lattice gas models of ionic superconductors. *J. Stat. Phys.*, 34:497, 1984.
- [56] M. R. Evans. Phase transitions in one-dimensional nonequilibrium systems. *Braz. J. Phys.*, 30:42–57, 2000.
- [57] C. T. MacDonald, J. H. Gibbs, and A. C Pipkin. Kinetics of biopolymerization on nucleic acid templates. *Biopolymers*, 6:1, 1968.
- [58] B. Derrida and K. Mallick. Exact diffusion constant for the one-dimensional partially asymmetric exclusion model. *J. Phys. A*, 30:1031, 1993.
- [59] B. Derrida and M.R. Evans. The asymmetric exclusion model: exact results through a matrix approach. In Privman [1], pages 277–304.
- [60] C. Boutillier, P. Francois, K. Mallick, and S. Mallick. A matrix ansatz for the diffusion of an impurity in the asymmetric simple exclusion process. *J. Phys. A*, 35:9703, 2002.
- [61] A. B. Kolomeisky. Exact solutions for a partially asymmetric exclusion model with two species. *Physica A*, 245:523, 1997.
- [62] C. Arita. Matrix product ground states for exclusion processes with parallel dynamics. *J. Stat. Mech.: Theor. Exp.*, page 12008, 2008.
- [63] N. Rajewsky. *Exact results for one-dimensional stochastic processes*. PhD thesis, Universität zu Köln, 1997.
- [64] M. R. Evans, S. N. Majumdar, and R. K. P. Zia. Factorized steady states in mass transport models. *J. Phys. A*, 37:275–280, 2004.
- [65] R. K. P. Zia, M. R. Evans, and S. N. Majumdar. Construction of the factorized steady state distribution in models of mass transport. *J. Stat. Mech.: Theor. Exp.*, page 10001, 2004.
- [66] S. Grosskinsky, G. M. Schuetz, and H. Spohn. Condensation in the zero range process: stationary and dynamical properties. *J. Stat. Phys.*, 113:389–410, 2003.
- [67] S. Grosskinsky and G. M. Schuetz. Discontinuous condensation transition and nonequivalence of ensembles in a zero-range process. *J. Stat. Phys.*, 132:77–108, 2008.
- [68] S. Grosskinsky and G. M. Schuetz. Discontinuous condensation transition and nonequivalence of ensembles in a zero-range process (erratum). *J. Stat. Phys.*, 133:801–803, 2008.
- [69] S. N. Majumdar, M. R. Evans, and R. K. P. Zia. The nature of the condensate in mass transport models. *Phys. Rev. Lett.*, 94:180601, 2005.
- [70] M. R. Evans, S. N. Majumdar, and R. K. P. Zia. Canonical analysis of condensation in factorized steady states. *J. Stat. Phys.*, 123:357, 2006.
- [71] E. Levine, G. Ziv, L. Gray, and D. Mukamel. Traffic jams and ordering far from thermal equilibrium. *Physica A*, 340:636, 2004.

- [72] K. Klauck and A. Schadschneider. On the ubiquity of matrix product states in one-dimensional stochastic models with boundary interactions. *Physica A*, 271:102, 1999.
- [73] L. De Sanctis and M. Isopi. On the matrix approach to interacting particle systems. *J. Stat. Phys.*, 115:383–393, 2004.
- [74] K. Krebs. On matrix product states for periodic boundary conditions. *J. Phys. A*, 33:149–154, 2000.
- [75] K. Krebs and S. Sandow. Matrix product eigenstates for one-dimensional stochastic models and quantum spin chains. *J. Phys. A*, 30:3165–3173, 1997.
- [76] R. J. Baxter. *Exactly solved models in statistical mechanics*. Academic Press Inc., London, 1982.
- [77] F. Essler and V. Rittenberg. Representations of the quadratic algebra and partially asymmetric diffusion with open boundaries. *J. Phys. A*, 29:3375–3407, 1996.
- [78] H. S. Wilf. *Generatingfunctionology*. Academic Press, San Diego, 2006.
- [79] J. de Gier. Exact stationary state for a deterministic high-speed traffic model with open boundaries. *J. Phys. A*, 34:3707–3720, 2001.
- [80] S. G. Mohanty. *Lattice path counting and applications*. Academic Press, London, 1979.
- [81] R. Brak and J. W. Essam. Asymmetric exclusion model and weighted lattice paths. *J. Phys. A*, 37:4183–4217, 2000.
- [82] R. A Blythe, W. Janke, D. A. Johnston, and R. Kenna. The grand-canonical asymmetric exclusion process and the one-transit walk. *J. Stat. Mech.: Theor. Exp.*, page 06001, 2004.
- [83] R. A Blythe, W. Janke, D. A. Johnston, and R. Kenna. Dyck paths, motzkin paths and traffic jams. *J. Stat. Mech.: Theor. Exp.*, page 10007, 2004.
- [84] R. K. P. Zia and B. Schmittmann. Probability currents as principal characteristics in the statistical mechanics of non-equilibrium steady states. *J. Stat. Mech.: Theor. Exp.*, page 07012, 2007.
- [85] M. Woelki. in preparation. 2009.
- [86] T. Antal and G. M. Schütz. Asymmetric exclusion process with next-nearest-neighbor interaction: Some comments on traffic flow and a nonequilibrium reentrance transition. *Phys. Rev. E*, 62:83–89, 2000.
- [87] W. Feller. *An Introduction to Probability Theory and Its Applications*. Wiley Series, 1968.
- [88] B. D. Hughes. *Random walks and random environments*. Oxford University Press, Oxford, 1996.
- [89] A. P. Isaev, P. N. Pyatov, and V. Rittenberg. Diffusion algebras. *J. Phys. A*, 34:5815–5834, 2001.

- [90] A. Schadschneider and M. Schreckenberg. Car-oriented mean-field theory for traffic flow models. *J. Phys. A*, 30:L69, 1997.
- [91] R. Rajesh and S. N. Majumdar. Conserved mass models and particle systems in one dimension. *J. Stat. Phys.*, 99:943, 2000.
- [92] L. Gray and D. Griffeath. The ergodic theory of traffic jams. *J. Stat. Phys.*, 105:413–451, 2001.
- [93] M. Fukui and Y. Ishibashi. Traffic flow in 1d cellular automaton model including cars moving with high speed. *Phys. Rev. E*, 51:2339, 1995.
- [94] M. Fukui, H. Oikawa, and Y. Ishibashi. Flow of cars crossing with unequal velocities in a two-dimensional cellular automaton model. *J. Phys. Soc. Jpn.*, 65:2514–2517, 1996.
- [95] M. Fukui and Y. Ishibashi. Effect of delay in restarting of stopped cars in a one-dimensional traffic model. *J. Phys. Soc. Jpn.*, 66:385–387, 1997.
- [96] B. H. Wang, Y.R. Kwong, and P.M. Hui. Statistical mechanical approach to fukui-ishibashi traffic flow model. *Phys. Rev. E*, 57:2568, 1998.
- [97] E. Levine, D. Mukamel, and G. M. Schuetz. Zero-range process with open boundaries. *J. Stat. Phys.*, 120:759, 2005.
- [98] F. H. Jafarpour and B. Ghavami. The discontinuous phase transition for an exactly solvable one-dimensional creation annihilation system. *J. Stat. Mech.*, page 08010, 2007.
- [99] F. Tabatabaei and G. M. Schuetz. Non-equilibrium field-induced phase separation in single-file diffusion. *Diffusion Fundamentals*, 4:5.1–5.38, 2006.

Zusammenfassung

Die vorliegende Arbeit beschäftigt sich im Wesentlichen mit dem vollkommen asymmetrischen Exklusionsprozess (ASEP) [4] mit periodischen Randbedingungen. Der ASEP ist auf einem eindimensionalen Gitter definiert, auf dem jede Zelle entweder leer oder von genau einem Teilchen besetzt sein kann. In der ursprünglichen Definition können sich die Teilchen unter der Voraussetzung, dass die Zielzelle frei ist, genau um eine Zelle nach rechts bewegen. Hier betrachten wir die Möglichkeit, eine oder zwei Zellen zu hüpfen. Sowohl zufällig-sequentielle als auch parallele Dynamik werden behandelt. Für einen speziellen Grenzfall, wo Teilchen unter Berücksichtigung des Abstand zum nächsten Teilchen und ihrer Maximalgeschwindigkeit sich immer so weit wie möglich vorwärts bewegen, entwickelt sich der Prozess in spezielle Unterräume des Konfigurationsraumes. Das ist es, was wir als ASEP mit Ausbildung von Überschussmasse bezeichnen: Die Zahl der freien Zellen zwischen den Teilchen ist immer ein Vielfaches der Maximalgeschwindigkeit zwei. Jedoch kann sich, abhängig von Teilchenzahl und Gittergröße, ein einzelnes überschüssiges Loch herausbilden. Das Interesse gilt der Berechnung des stationären Zustands im Bezugsrahmen dieses überschüssigen Lochs. Man beobachtet zwei Phasen, in denen seine Geschwindigkeit unterschiedliche Werte annimmt und als Ordnungsparameter für den Übergang erster Ordnung dient. Für kontinuierliche Zeit steht der Prozess in Zusammenhang mit dem ASEP und einem Defektteilchen [23] und für parallele Dynamik erzeugt er eine neue natürliche Defektdynamik: In Phase 1 haben die typischen Dichteprofile exponentiell gedämpfte Oszillationen um den Defekt und erreichen in ausreichender Entfernung einen konstanten Wert. In Phase 2 ist das Profil algebraisch: Vor dem Defekt fällt es von einem erhöhten Wert langsam ab und ist hinter ihm entsprechend erniedrigt. Dies ist ein typischer Grenzfall eines Schockprofils mit gleicher Dichte weit links wie rechts vom Defekt.

Der mathematische Rahmen zur Formulierung der exakten Resultate dieses und verwandter Prozesse ist der Matrixproduktansatz [15], welcher in verschiedene Richtungen erweitert wird. Ein Ausgangspunkt der Arbeit war die exakte Lösbarkeit ASEP-artiger Modelle und der Fokus ist immer wieder auf der Lösung selbst. Als Nebenprodukt erhalten wir, dass die stationären Gewichte des ASEP mit offenen Rändern und paralleler Dynamik als Produkt aus einem paarfaktorisierten Term und einem Matrixprodukt geschrieben werden können. Dies ersetzt die bekannte Matrix-in-Matrix-Form [24] und suggeriert, dass es einen generellen Zusammenhang zwischen Matrixproduktzuständen für zufällig-sequentielle, verschiedene zeitdiskrete [13] und parallele Dynamik gibt.

Für den ASEP mit Maximalgeschwindigkeit zwei und etwas allgemeinerer Definition erhalten wir den exakten stationären Zustand für zwei Teilchen auf dem Ring. Dieses Resultat zeigt, dass es generell Oszillationen in der Teilchenabstandsverteilung geben kann. Eine Mean-Field-Theorie, welche Korrelationen aufeinanderfolgender Teilchen berücksichtigt, zeigt thermodynamisch eine bemerkenswerte Übereinstimmung mit Computersimulationen. Für den ASEP mit Ausbildung von Überschussmasse sagt die Theorie die Abstandsverteilung exakt voraus, da der einzelne Defekt darauf thermodynamisch keinen Einfluss nimmt. Diese Beobachtung sollte auf eine breitere Klasse von Modellen zutreffen und darüberhinaus erlauben, exakte Aussagen zu machen, wenn den Modellen kein Produktmaß zugrunde liegt.

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Erklärung

Ich versichere, dass ich die von mir vorgelegte Dissertation selbständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit – einschließlich der Abbildungen –, die anderen Werken im Wortlaut oder dem Sinn nach entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie – abgesehen von unten angegebenen Teilpublikationen – noch nicht veröffentlicht worden ist sowie, dass ich eine solche Veröffentlichung vor Abschluß des Promotionsverfahrens nicht vornehmen werde. Die Bestimmungen dieser Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Herrn Professor Dr. Michael Schreckenberg betreut worden.

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